# Reasoning about Reliability and Cost using Decision Diagrams and Syntax Trees


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# Motivation

- Reliability and availability are important aspects of system design
- Many life-critical systems are required to operate without a system failure for a given period of time (nuclear, aerospace, spacecraft) - *reliability*
- Fraction of time the system is providing service to its users - *availability*
- We want to design systems as reliable (available) as possible. But, higher the reliability - higher the *cost*. This is a *multi-criteria optimization problem*.
- We often need real-time answers (e.g. in an interactive user-centric design tool)
- We suggest a generic method for *real-time interactive* multi-criteria optimization involving reliability or availability as one of the criteria

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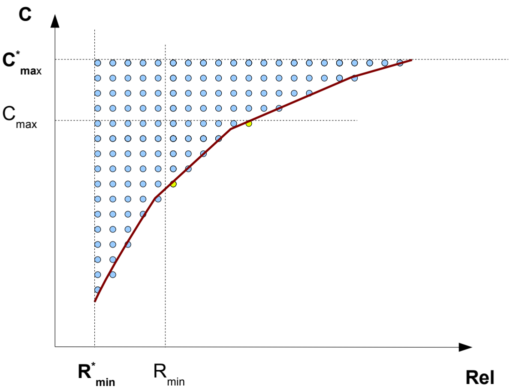
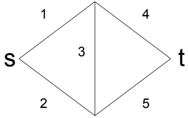
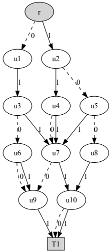
We generate *efficient frontier* in the offline phase to support efficient online interaction



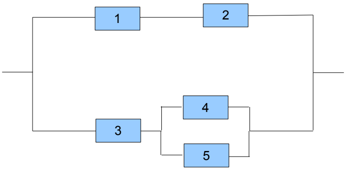
We handle reliability functions modeled as *probability graphs* by using *decision diagrams*

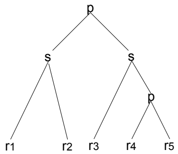



We introduce several algorithmic schemes that exploit specific aspects of decision diagrams to enhance efficient frontier generation

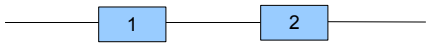
We develop a specialized approach when reliability function is specified as a *block diagram*. We introduce the notion of *syntax trees* to exploit internal structure of block diagrams for faster frontier generation



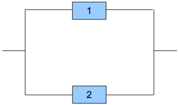




We demonstrate through experiments the value of our approach

- Computing reliability and availability requires the same computational mechanism.
- Reliability of the $i$-th component expressed as a value $r_i \in [0, 1]$.
  - $r_i = 0$ - component not available (not functional)
  - $r_i = 1$ - component available (functional)
  - $r_i = 0.999$ - component available 99.9% of the time (functional with probability 99.9%)
- Two fundamental connection types for components $r_1, r_2$:
  - *serial*, $Rel(r_1, r_2) = r_1 \cdot r_2$




  - *parallel*, $Rel(r_1, r_2) = 1 - (1 - r_1) \cdot (1 - r_2)$

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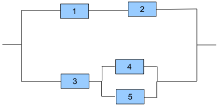
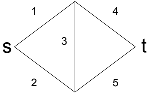
# Block Diagrams and Probability Graphs

- *Block diagrams* consist only of parallel and serial connections and can be *efficiently* evaluated.




- *Probability graphs* are general graphs with edges labeled with probabilities. Reliability corresponds to a probability of existence of a path between a source and a terminal. Computing this probability is *NP-hard* (Valiant, 1979)[1]. State-of-the-art approach is based on *decision diagrams*.




[1] The complexity of enumeration and reliability problems, SIAM J. Comput.

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# Decision Diagrams

**Definition (Multivalued Decision Diagrams)**

A *decision diagram* is a rooted directed acyclic graph $G = (V, E)$ where every node $u$ is labeled with a variable $x_i$ and every edge $e$, originating from a node labeled $x_i$, is labeled with a value $a_i \in D_i$. The decision diagram contains a special *terminal* node **1**, that has no outgoing edges.

- Every path from root to terminal encodes an assignment
- MDDs are almost always assumed to be *ordered*
- MDDs achieve exponential space savings by *merging isomorphic nodes*
- MDDs can achieve additional savings by *removing redundant nodes*
- If all domains $D_i$ are binary, i.e. $D_1 = \ldots = D_n = \{0, 1\}$, then we have a *binary decision diagram* (BDD)

# Decision Diagrams

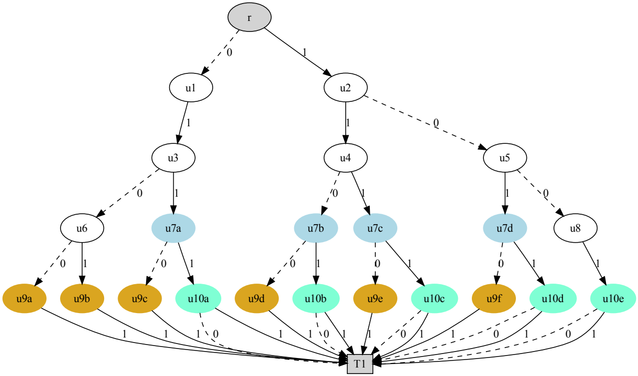



Figure: Expanded MDD for a Boolean function $(x_1 \wedge x_4) \vee (x_2 \wedge x_5) \vee (x_1 \wedge x_3 \wedge x_5) \vee (x_2 \wedge x_3 \wedge x_4)$

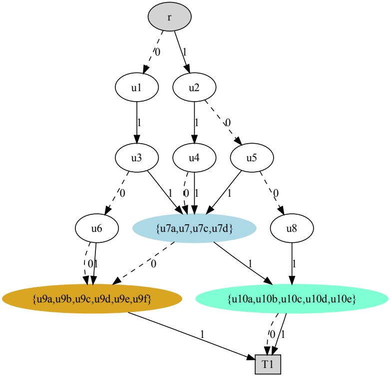



Figure: Merged MDD for a Boolean function $(x_1 \wedge x_4) \vee (x_2 \wedge x_5) \vee (x_1 \wedge x_3 \wedge x_5) \vee (x_2 \wedge x_3 \wedge x_4)$

# Decision Diagrams

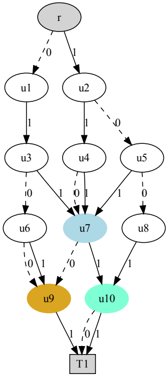



Figure: Merged MDD for a Boolean function $(x_1 \wedge x_4) \vee (x_2 \wedge x_5) \vee (x_1 \wedge x_3 \wedge x_5) \vee (x_2 \wedge x_3 \wedge x_4)$

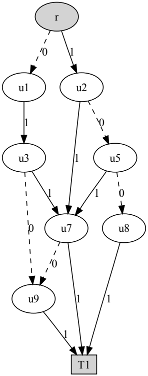
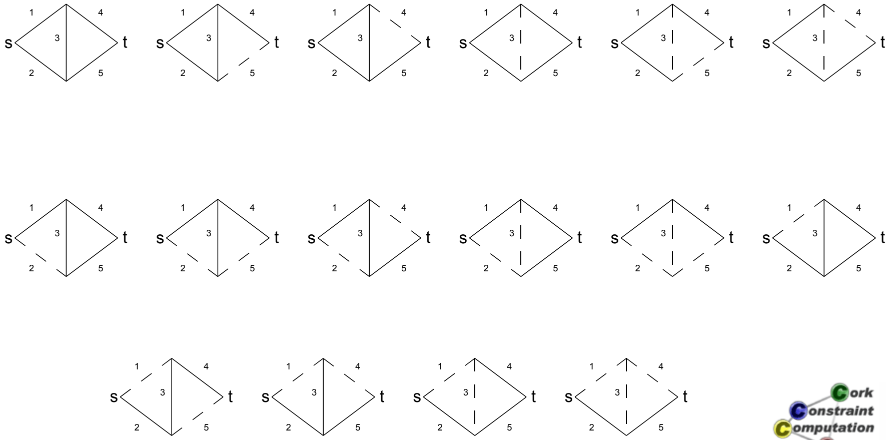


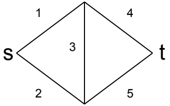

Figure: Reduced MDD for a Boolean function $(x_1 \wedge x_4) \vee (x_2 \wedge x_5) \vee (x_1 \wedge x_3 \wedge x_5) \vee (x_2 \wedge x_3 \wedge x_4)$

Probability of existence of an $s - t$ path is the probability of the graph having one of the following 16 configurations:

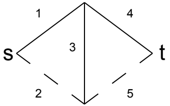


# Computing Reliability of Probabilistic Graphs

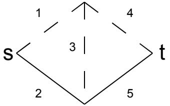
Each graph configuration has a probability of occurrence.




$$P(e_1 = 1, e_2 = 1, e_3 = 1, e_4 = 1, e_5 = 1) = r_1 \cdot r_2 \cdot r_3 \cdot r_4 \cdot r_5$$




$$P(e_1 = 1, e_2 = 0, e_3 = 1, e_4 = 1, e_5 = 0) = r_1 \cdot (1 - r_2) \cdot r_3 \cdot r_4 \cdot (1 - r_5)$$




$$P(e_1 = 0, e_2 = 1, e_3 = 0, e_4 = 0, e_5 = 1) = (1 - r_1) \cdot r_2 \cdot (1 - r_3) \cdot (1 - r_4) \cdot r_5$$

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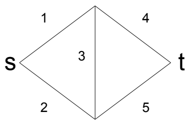
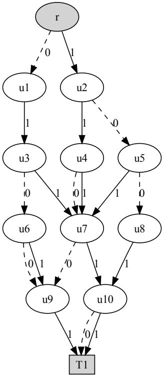
Probability of occurrence of one of the feasible graph configurations is the sum of the probabilities of each configuration.

$$
\begin{aligned}
P_{st} = \quad & P(e_1 = 1, e_2 = 1, e_3 = 1, e_4 = 1, e_5 = 1) + \\
& P(e_1 = 1, e_2 = 1, e_3 = 1, e_4 = 1, e_5 = 0) + \\
& \ldots \\
& P(e_1 = 0, e_2 = 1, e_3 = 0, e_4 = 0, e_5 = 1)
\end{aligned}
$$

We compute a BDD representing the set of all feasible assignments.

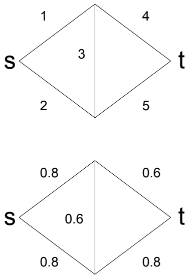


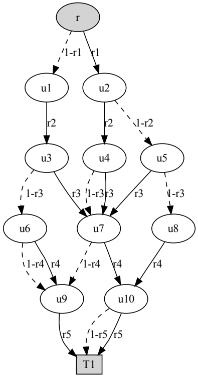

| $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 |

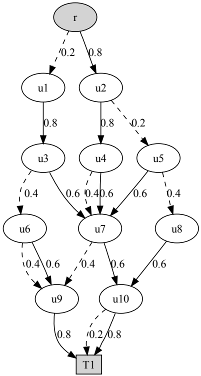


# Computing Reliability of Probabilistic Graphs

For given edge reliabilities $r_1, \ldots, r_n$ we label BDD edges correspondingly: $e_i = 1$ edge with $r_i$ and $e_i = 0$ edge with $1 - r_i$




Use recursive evaluation over BDD (linear in BDD size): $Rel(\mathbf{1}) = 1$, $Rel(u) = \sum_{e:u\rightarrow u'} r_e \cdot Rel(u')$.

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# Network Design Problem

- In previous example we *evaluate* reliability function *Rel* for fixed values: $Rel(0.8, 0.8, 0.6, 0.6, 0.8) = 0.85504$. And this alone is an NP-hard problem!
- In a *network design problem* we have *multiple options* for each edge. Let $x_i \in D_i$ represent an implementation of edge $e_i$. For each implementation we have a *reliability* $r_i(x_i)$ and a *cost* $c_i(x_i)$. For a full assignment $x_1, \ldots, x_n$ we can compute reliability of the system $Rel(x_1, \ldots, x_n)$ and its (additive) cost $C(x_1, \ldots, x_n) = \sum_i c_i(x_i)$.
- We are interested in finding an implementation of the system $x_1 = v_1, \ldots, x_n = v_n$ that satisfies requirements for minimal reliability $R_{min}$ and maximal budget $C_{max}$

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# Multicriteria Queries

For a given reliability function $Rel(x_1, \ldots, x_n)$ and given cost function $C(x_1, \ldots, x_n) = \sum_i c_i(x_i)$ we consider the following queries:

**Q1 - Minimal Cost for a given Reliability**

$$min\ C(\overrightarrow{x}),\ \text{s.t. } Rel(\overrightarrow{x}) \geq R_{min}.$$

**Q2 - Maximal Reliability within a Budget**

$$max\ Rel(\overrightarrow{x}),\ \text{s.t. } C(\overrightarrow{x}) \leq C_{max}.$$

**Q3 - Feasibility of a Component** Given $x_i, v \in D_i$, is there a solution $\overrightarrow{x}$ containing $x_i = v$ s.t. $Rel(\overrightarrow{x}) \geq R_{min}$ and $C(\overrightarrow{x}) \leq C_{max}$.

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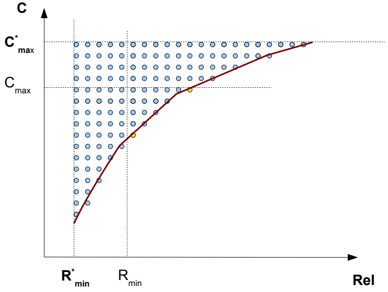
# Answering Q1 and Q2

We answer Q1 and Q2 through generating an *efficient frontier* - the set of all *nondominated* pairs $(Rel(x), C(x))$.

## Definition (Domination)

Pair $(Rel(\overrightarrow{x}_1), C(\overrightarrow{x}_1))$ *is dominated by* $(Rel(\overrightarrow{x}_2), C(\overrightarrow{x}_2))$, iff $Rel(\overrightarrow{x}_1) \leq Rel(\overrightarrow{x}_2) \wedge C(\overrightarrow{x}_1) \geq C(\overrightarrow{x}_2)$

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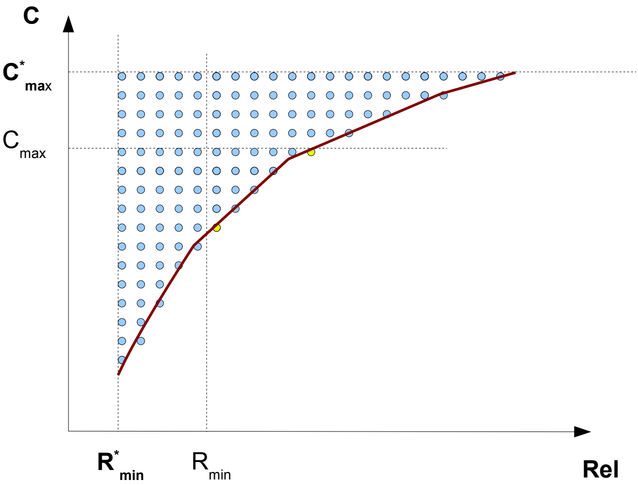
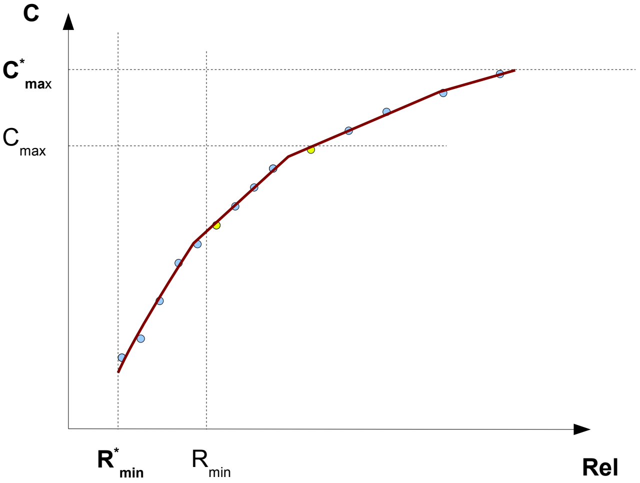
Efficient frontier and the dominated feasible assignments.



We compute only the efficient frontier.



# Efficient Frontier Generation for Probabilistic Graphs

- We generate efficient frontier through *exhaustive depth-first search*
- The algorithm takes the number of steps linear in the number of feasible (possibly dominated) solutions
- In the worst case we have to evaluate function $Rel(x_1, \ldots, x_n)$ (NP-Hard) exponential number of times.
- Do we have to solve an NP-hard problem every time we evaluate $Rel(x_1, \ldots, x_n)$?

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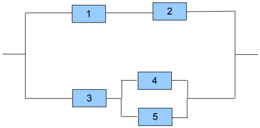
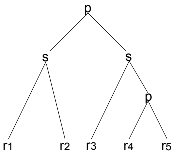
- The answer is *no*.
- We use a BDD representation of the reliability function. We generate BDD encoding of graph topology *only once*, and then repeatedly apply different labels $r_1(x_1), \ldots, r_n(x_n)$.

- Furthermore, we can do this *incrementally* by restricting DFS branching order to the *BDD variable ordering* and using *intelligent caching* techniques over BDD nodes
- We never explore infeasible partial assignments: we exploit *monotonicity of reliability function*

# Efficient Frontier Generation for Block Diagrams

- For a reliability function given as a block diagram we generate efficient frontier over *syntax trees*




- We initialize efficient lists at each leaf node. For a leaf node $u_i$ corresponding to $x_i$ we set $L = \{(r_i(v), c_i(v)) | v \in D_i\}$
- We aggregate lists at each internal node. For a node $u$ with a left child list $L_l$ and right child list $L_r$ we set $L \leftarrow L_l \otimes_u L_r$
- $(r_1, c_1) \otimes_u (r_2, c_2) = (r_1 \otimes_u r_2, c_1 + c_2)$. For a serial node $r_1 \otimes_u r_2 = r_1 \cdot r_2$. For a parallel node $r_1 \otimes_u r_2 = 1 - (1 - r_1) \cdot (1 - r_2)$.

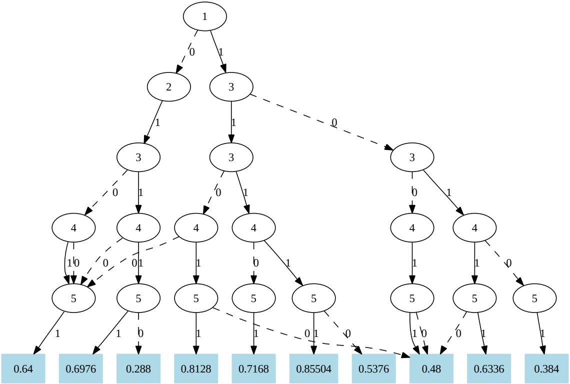
| Vars | Avg | | | Max | | |
|---|---|---|---|---|---|---|
| | $t_{rbd}$ | $t_{mdd}$ | $t^{0.9}_{mdd}$ | $t_{rbd}$ | $t_{mdd}$ | $t^{0.9}_{mdd}$ |
| 20 | 0.0003 | 0.037 | 0.008 | 0.01 | 0.54 | 0.51 |
| 23 | 0.0006 | 0.756 | 0.137 | 0.01 | 4.47 | 3.58 |
| 24 | 0.0007 | 1.295 | 0.22 | 0.01 | 8.99 | 7.57 |
| 25 | 0.0008 | 3.769 | 0.29 | 0.01 | 23.69 | 11.45 |
| 26 | 0.0010 | 5.902 | 0.36 | 0.01 | 31.59 | 31.23 |
| 27 | 0.0017 | 30.52 | 1.80 | 0.01 | 119.91 | 53.05 |
| 50 | 0.0048 | - | - | 0.02 | - | - |
| 75 | 0.0179 | - | - | 0.08 | - | - |
| 100 | 0.0594 | - | - | 0.57 | - | - |
| 250 | 1.3642 | - | - | 7.48 | - | - |

Table: Results over 1000 trials with a time cutoff for each run of 10 minutes. All times are in seconds. For instances with more than 50 variables no solutions were generated in the MDD-based approaches due to time-out.

# Beyond CRAGS: Answering Q3

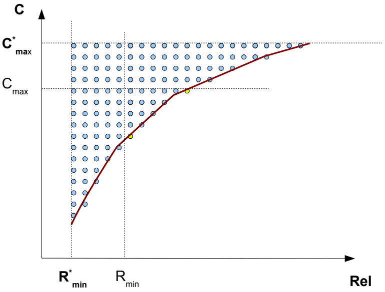
**Q3 - Feasibility of a Component** Given $x_i, v \in D_i$, is there a solution $\overrightarrow{x}$ containing $x_i = v$ s.t. $Rel(\overrightarrow{x}) \geq R_{min}$ and $C(\overrightarrow{x}) \leq C_{max}$.

We construct a *multi-terminal decision diagram* (MTMDD).



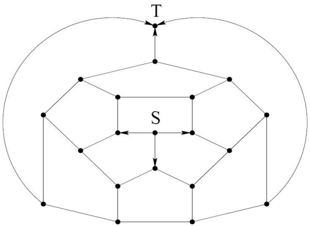
# Beyond CRAGS: MTMDD

Multi-terminal decision diagram represents non-dominated solutions as well.



# Beyond CRAGS: Experiments

Evaluated performance from a large instance from the literature, containing 20 nodes. Corresponding merged MDD has 2414 nodes and 4730 edges.



# Beyond CRAGS: Experiments

| Scenario | d | Distr | $r_{min}$ | Avg Term | Avg V | Avg E | Avg t |
|---|---|---|---|---|---|---|---|
| Wireless | $10^2$ | linear | 0.80 | 9 | 37,422 | 72,206 | 25.6s |
| | $10^3$ | linear | 0.80 | 95 | 78,716 | 147,271 | 25.7s |
| | $10^4$ | linear | 0.80 | 941 | 108,335 | 198,688 | 25.6s |
| Wired | 7 | expo | 0.95 | 4 | 22,505 | 44,494 | 2m 53s |
| | 7 | expo | 0.98 | 4 | 19,568 | 38,831 | 5m 31s |
| | 7 | expo | 0.99 | 4 | 19,099 | 37,978 | 5m 37s |

Table: MTMDD compilation for the `Relex9` instance. The results are averaged over 100 trials, with average and maximal values encountered shown in the table. Column **Term** indicates the number of terminals. Column **t** indicates computation time.