Reasoning about Reliability and Cost using Decision Diagrams and Syntax Trees

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Motivation

- Reliability and availability are important aspects of system design.
- Many life-critical systems are required to operate without a system failure for a given period of time (nuclear, aerospace, spacecraft) - *reliability*.
- Fraction of time the system is providing service to its users - *availability*.
- We want to design systems as reliable (available) as possible. But, higher the reliability - higher the *cost*. This is a *multi-criteria optimization problem*.
- We often need real-time answers (e.g. in an interactive user-centric design tool).
- We suggest a generic method for *real-time interactive* multi-criteria optimization involving reliability or availability as one of the criteria.
Overview of our Approach

We generate *efficient frontier* in the offline phase to support efficient online interaction.
Overview of our Approach

We handle reliability functions modeled as *probability graphs* by using *decision diagrams*

We introduce several algorithmic schemes that exploit specific aspects of decision diagrams to enhance efficient frontier generation.
Overview of our Approach

We develop a specialized approach when reliability function is specified as a *block diagram*. We introduce the notion of *syntax trees* to exploit internal structure of block diagrams for faster frontier generation.

We demonstrate through experiments the value of our approach.
Computing reliability and availability requires the same computational mechanism.

Reliability of the $i$-th component expressed as a value $r_i \in [0, 1]$.

- $r_i = 0$ - component not available (not functional)
- $r_i = 1$ - component available (functional)
- $r_i = 0.999$ - component available 99.9% of the time (functional with probability 99.9%)

Two fundamental connection types for components $r_1, r_2$:

- **serial**, $Rel(r_1, r_2) = r_1 \cdot r_2$

  ![Serial Connection Diagram]

- **parallel**, $Rel(r_1, r_2) = 1 - (1 - r_1) \cdot (1 - r_2)$

  ![Parallel Connection Diagram]
Block diagrams consist only of parallel and serial connections and can be efficiently evaluated.

Probability graphs are general graphs with edges labeled with probabilities. Reliability corresponds to a probability of existence of a path between a source and a terminal. Computing this probability is \textit{NP-hard} (Valiant, 1979)\textsuperscript{1}. State-of-the-art approach is based on decision diagrams.

\textsuperscript{1}The complexity of enumeration and reliability problems, \textit{SIAM J. Comput.}
Definition (Multivalued Decision Diagrams)

A decision diagram is a rooted directed acyclic graph $G = (V, E)$ where every node $u$ is labeled with a variable $x_i$ and every edge $e$, originating from a node labeled $x_i$, is labeled with a value $a_i \in D_i$. The decision diagram contains a special terminal node $1$, that has no outgoing edges.

- Every path from root to terminal encodes an assignment
- MDDs are almost always assumed to be ordered
- MDDs achieve exponential space savings by merging isomorphic nodes
- MDDs can achieve additional savings by removing redundant nodes
- If all domains $D_i$ are binary, i.e. $D_1 = \ldots = D_n = \{0, 1\}$, then we have a binary decision diagram (BDD)
Figure: Expanded MDD for a Boolean function
\((x_1 \land x_4) \lor (x_2 \land x_5) \lor (x_1 \land x_3 \land x_5) \lor (x_2 \land x_3 \land x_4)\)
Figure: Merged MDD for a Boolean function

\((x_1 \land x_4) \lor (x_2 \land x_5) \lor (x_1 \land x_3 \land x_5) \lor (x_2 \land x_3 \land x_4)\)
Figure: Merged MDD for a Boolean function

\((x_1 \land x_4) \lor (x_2 \land x_5) \lor (x_1 \land x_3 \land x_5) \lor (x_2 \land x_3 \land x_4)\)
Figure: Reduced MDD for a Boolean function

\[(x_1 \land x_4) \lor (x_2 \land x_5) \lor (x_1 \land x_3 \land x_5) \lor (x_2 \land x_3 \land x_4)\]
Probability of existence of an $s-t$ path is the probability of the graph having one of the following 16 configurations:
Each graph configuration has a probability of occurrence.

\[ P(e_1 = 1, e_2 = 1, e_3 = 1, e_4 = 1, e_5 = 1) = r_1 \cdot r_2 \cdot r_3 \cdot r_4 \cdot r_5 \]

\[ P(e_1 = 1, e_2 = 0, e_3 = 1, e_4 = 1, e_5 = 0) = r_1 \cdot (1 - r_2) \cdot r_3 \cdot r_4 \cdot (1 - r_5) \]

\[ P(e_1 = 0, e_2 = 1, e_3 = 0, e_4 = 0, e_5 = 1) = (1 - r_1) \cdot r_2 \cdot (1 - r_3) \cdot (1 - r_4) \cdot r_5 \]
Probability of occurrence of one of the feasible graph configurations is the sum of the probabilities of each configuration.

\[
P_{st} = P(e_1 = 1, e_2 = 1, e_3 = 1, e_4 = 1, e_5 = 1) + P(e_1 = 1, e_2 = 1, e_3 = 1, e_4 = 1, e_5 = 0) + \ldots + P(e_1 = 0, e_2 = 1, e_3 = 0, e_4 = 0, e_5 = 1)
\]
We compute a BDD representing the set of all feasible assignments.

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![BDD Diagram](image-url)
For given edge reliabilities $r_1, \ldots, r_n$ we label BDD edges correspondingly: $e_i = 1$ edge with $r_i$ and $e_i = 0$ edge with $1 - r_i$.

Use recursive evaluation over BDD (linear in BDD size): $\text{Rel}(1) = 1$, $\text{Rel}(u) = \sum_{e: u \rightarrow u'} r_e \cdot \text{Rel}(u')$. 

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In previous example we evaluate reliability function $Rel$ for fixed values: $Rel(0.8, 0.8, 0.6, 0.6, 0.8) = 0.85504$. And this alone is an NP-hard problem!

In a network design problem we have multiple options for each edge. Let $x_i \in D_i$ represent an implementation of edge $e_i$. For each implementation we have a reliability $r_i(x_i)$ and a cost $c_i(x_i)$. For a full assignment $x_1, \ldots, x_n$ we can compute reliability of the system $Rel(x_1, \ldots, x_n)$ and its (additive) cost $C(x_1, \ldots, x_n) = \sum_i c_i(x_i)$.

We are interested in finding an implementation of the system $x_1 = v_1, \ldots, x_n = v_n$ that satisfies requirements for minimal reliability $R_{\text{min}}$ and maximal budget $C_{\text{max}}$. 

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Reliability and Cost Multicriteria Reasoning
For a given reliability function \( \text{Rel}(x_1, \ldots, x_n) \) and given cost function \( C(x_1, \ldots, x_n) = \sum_i c_i(x_i) \) we consider the following queries:

Q1 - Minimal Cost for a given Reliability
\[
\min C(\vec{x}), \quad \text{s.t.} \quad \text{Rel}(\vec{x}) \geq R_{\text{min}}.
\]

Q2 - Maximal Reliability within a Budget
\[
\max \text{Rel}(\vec{x}), \quad \text{s.t.} \quad C(\vec{x}) \leq C_{\text{max}}.
\]

Q3 - Feasibility of a Component
Given \( x_i, v \in D_i \), is there a solution \( \vec{x} \) containing \( x_i = v \) s.t. \( \text{Rel}(\vec{x}) \geq R_{\text{min}} \) and \( C(\vec{x}) \leq C_{\text{max}} \).
Answering Q1 and Q2

We answer Q1 and Q2 through generating an efficient frontier - the set of all nondominated pairs \((\text{Rel}(x), \text{C}(x))\).

**Definition (Domination)**

Pair \((\text{Rel}(\vec{x}_1), \text{C}(\vec{x}_1))\) is dominated by \((\text{Rel}(\vec{x}_2), \text{C}(\vec{x}_2))\), iff \(\text{Rel}(\vec{x}_1) \leq \text{Rel}(\vec{x}_2) \land \text{C}(\vec{x}_1) \geq \text{C}(\vec{x}_2)\)
Answering Q1 and Q2

Efficient frontier and the dominated feasible assignments.
Answering Q1 and Q2

We compute only the efficient frontier.
We generate efficient frontier through *exhaustive depth-first search*

The algorithm takes the number of steps linear in the number of feasible (possibly dominated) solutions.

In the worst case we have to evaluate function $\text{Rel}(x_1, \ldots, x_n)$ (NP-Hard) exponential number of times.

Do we have to solve an NP-hard problem every time we evaluate $\text{Rel}(x_1, \ldots, x_n)$?
The answer is *no*.

We use a BDD representation of the reliability function. We generate BDD encoding of graph topology *only once*, and then repeatedly apply different labels $r_1(x_1), \ldots, r_n(x_n)$. 
Furthermore, we can do this *incrementally* by restricting DFS branching order to the *BDD variable ordering* and using *intelligent caching* techniques over BDD nodes.

We never explore infeasible partial assignments: we exploit *monotonicity of reliability function*. 
For a reliability function given as a block diagram we generate efficient frontier over syntax trees.

We initialize efficient lists at each leaf node. For a leaf node $u_i$ corresponding to $x_i$ we set $L = \{(r_i(v), c_i(v)) | v \in D_i\}$

We aggregate lists at each internal node. For a node $u$ with a left child list $L_l$ and right child list $L_r$ we set $L \leftarrow L_l \otimes_u L_r$

$(r_1, c_1) \otimes_u (r_2, c_2) = (r_1 \otimes_u r_2, c_1 + c_2)$. For a serial node $r_1 \otimes_u r_2 = r_1 \cdot r_2$. For a parallel node $r_1 \otimes_u r_2 = 1 - (1 - r_1) \cdot (1 - r_2)$. 

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Table: Results over 1000 trials with a time cutoff for each run of 10 minutes. All times are in seconds. For instances with more than 50 variables no solutions were generated in the MDD-based approaches due to time-out.
Q3 - Feasibility of a Component  Given $x_i, v \in D_i$, is there a solution $\vec{x}$ containing $x_i = v$ s.t. $Rel(\vec{x}) \geq R_{min}$ and $C(\vec{x}) \leq C_{max}$.

We construct a multi-terminal decision diagram (MTMDD).
Multi-terminal decision diagram represents non-dominated solutions as well.
Evaluated performance from a large instance from the literature, containing 20 nodes. Corresponding merged MDD has 2414 nodes and 4730 edges.
<table>
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<tr>
<th>Scenario</th>
<th>d</th>
<th>Distr</th>
<th>r_{\text{min}}</th>
<th>Term</th>
<th>V</th>
<th>E</th>
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**Table:** MTMDD compilation for the Relex9 instance. The results are averaged over 100 trials, with average and maximal values encountered shown in the table. Column **Term** indicates the number of terminals. Column **t** indicates computation time.