Chapter 4: Basic Constraint Reasoning
(SEND+MORE=MONEY)

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ECLiPSe ELearning
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Outline

1. Problem
2. Program
3. Constraint Setup
4. Search
5. Lessons Learned
What we want to introduce

- Finite Domain Solver in ECLiPSe
- Models and Programs
- Constraint Propagation and Search
- Basic constraints: linear arithmetic, alldifferent, disequality
- Built-in search: Labeling
- Visualizers for variables, constraints and search
Outline

1. Problem
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A Crypt-Arithmetic Puzzle

We begin with the definition of the SEND+MORE=MONEY puzzle. It is often shown in the form of a hand-written addition:

```
    S E N D
   + M O R E
   ________
    M O N E Y
```
Rules

- Each character stands for a digit from 0 to 9.
- Numbers are built from digits in the usual, positional notation.
- Repeated occurrence of the same character denote the same digit.
- Different characters denote different digits.
- Numbers do not start with a zero.
- The equation must hold.

\[
\begin{array}{c}
    S & E & N & D \\
+  & M & O & R & E \\
\hline
    M & O & N & E & Y
\end{array}
\]
Outline

1. Problem
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Each character is a variable, which ranges over the values 0 to 9.

An *alldifferent* constraint between all variables, which states that two different variables must have different values. This is a very common constraint, which we will encounter in many other problems later on.

Two *disequality constraints* (variable $X$ must be different from value $V$) stating that the variables at the beginning of a number can not take the value 0.

An arithmetic *equality constraint* linking all variables with the proper coefficients and stating that the equation must hold.
:- module(sendmory).

 Define Module

:- export(sendmory/1).

:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    S \= 0, M \= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
³ Make predicate visible
:- lib(ic).

sendmory(L):-
L = [S,E,N,D,M,O,R,Y],
L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D + 
    1000*M + 100*O + 10*R + E != 
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

Use ic library

sendmory(L):-
L = [S,E,N,D,M,O,R,Y],
L :: 0..9,

alldifferent(L),
S #\= 0, M #\= 0,
1000*S + 100*E + 10*N + D +
1000*M + 100*O + 10*R + E #= 
10000*M + 1000*O + 100*N + 10*E + Y,

labeling(L).
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L) :-

Predicate definition
L = [S,E,N,D,M,O,R,Y],
L :: 0..9,
\textbf{alldifferent} (L),
S #\neq 0, M #\neq 0,
1000 \times S + 100 \times E + 10 \times N + D +
1000 \times M + 100 \times O + 10 \times R + E \neq
10000 \times M + 1000 \times O + 100 \times N + 10 \times E + Y,
\textbf{labeling} (L).
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],  \ Define list
    L :: 0..9,
    alldifferent(L),
    S \= 0, M \= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
   L = [S,E,N,D,M,O,R,Y],
   L :: 0..9, % Define integer domain 0..9
   alldifferent(L),
   S #\= 0, M #\= 0,
   1000*S + 100*E + 10*N + D +
   1000*M + 100*O + 10*R + E #=
   10000*M + 1000*O + 100*N + 10*E + Y,
   labeling(L).
Program Sendmory

:- module (sendmory).
:- export (sendmory/1).
:- lib (ic).

sendmory (L):-
  L = [S,E,N,D,M,O,R,Y],
  L :: 0..9,
  alldifferent (L),  \( Digits \text{ must be different} \)
  S #\= 0, M #\= 0,
  1000*S + 100*E + 10*N + D +
  1000*M + 100*O + 10*R + E #=
  10000*M + 1000*O + 100*N + 10*E + Y,\n  labeling (L).
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
L = [S,E,N,D,M,O,R,Y],
L :: 0..9,
alldifferent(L),
S #\= 0, M #\= 0, Numbers don't start with 0
1000*S + 100*E + 10*N + D +
1000*M + 100*O + 10*R + E #= 
10000*M + 1000*O + 100*N + 10*E + Y,
labeling(L).
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
  L = [S,E,N,D,M,O,R,Y],
  L :: 0..9,
  alldifferent(L),
  S #\= 0, M #\= 0,
  1000*S + 100*E + 10*N + D +
  1000*M + 100*O + 10*R + E #=
  10000*M + 1000*O + 100*N + 10*E + Y,
  labeling(L).

$S E N D$
$+ M O R E$
$\rightarrow M O N E Y$
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #= 
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).\* built-in search routine
:- module(sendmory).
:- export(sendmory/1).

Export and
:- lib(ic).

sendmory(L):-

definition must match
L = [S,E,N,D,M,O,R,Y],
L :: 0..9,

alldifferent(L),
S \= 0, M \= 0,
1000*S + 100*E + 10*N + D +
1000*M + 100*O + 10*R + E #=
10000*M + 1000*O + 100*N + 10*E + Y,
labeling(L)._
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L) :- for predicate definition
  L = [S,E,N,D,M,O,R,Y],
  L :: 0..9,
  alldifferent(L),
  S \= 0, M \= 0,
  1000*S + 100*E + 10*N + D +
  1000*M + 100*O + 10*R + E #= 
  10000*M + 1000*O + 100*N + 10*E + Y,
  labeling(L).
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
  L = [S,E,N,D,M,O,R,Y],
  L :: 0..9,
  alldifferent(L),
  S #\= 0, M #\= 0,"Special symbol for ic"
  1000*S + 100*E + 10*N + D +
  1000*M + 100*O + 10*R + E #=
  10000*M + 1000*O + 100*N + 10*E + Y,
  labeling(L).
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],  \(\Rightarrow\) Confusing name!
    L :: 0..9,
    alldifferent(L),
    S \#\= 0, M \#\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E \#= 
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
Program Sendmory

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #= 
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
General Program Structure

:\- module(sendmory).
:\- export(sendmory/1).
:\- lib(ic).
sendmory(L):-
    L = [S,E,N,D,M,O,R,Y], VARIABLEs
    L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #= 
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
:- module(sendlmory).
:- export(sendlmory/1).
:- lib(ic).

sendlmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    Constraints
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
General Program Structure

:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
  L = [S,E,N,D,M,O,R,Y],
  L :: 0..9,
  alldifferent(L),
  S #\= 0, M #\= 0,
  1000*S + 100*E + 10*N + D +
  1000*M + 100*O + 10*R + E #=
  10000*M + 1000*O + 100*N + 10*E + Y,
  labeling(L).  =>  Search
Choice of Model

- This is *one* model, not *the* model of the problem
- Many possible alternatives
- Choice often depends on your constraint system
  - Constraints available
  - Reasoning attached to constraints
- Not always clear which is the *best* model
- Often: Not clear what is the *problem*

Alternative 1  Alternative 2
Running the program

- To run the program, we have to enter the query
  - `sendmory:sendmory(L)`.

- Result
  - \( L = [9, 5, 6, 7, 1, 0, 8, 2] \)
  - yes (0.00s cpu, solution 1, maybe more)
But how did the program come up with this solution?
Outline

1. Problem
2. Program
3. Constraint Setup
   - Domain Definition
   - Alldifferent Constraint
   - Disequality Constraints
   - Equality Constraint
4. Search
5. Lessons Learned
Domain Definition

\[ L = \{ S, E, N, D, M, O, R, Y \}, \]
\[ L :: 0..9, \]
\[ \{ S, E, N, D, M, O, R, Y \} \subseteq \{ 0..9 \} \]
# Domain Visualization

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Domain Visualization

\[
\begin{array}{cccccccccc}
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S &   &   &   &   &   &   &   &   &   \\
E &   &   &   &   &   &   &   &   &   \\
N &   &   &   &   &   &   &   &   &   \\
D &   &   &   &   &   &   &   &   &   \\
M &   &   &   &   &   &   &   &   &   \\
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\end{array}
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Rows = Variables
## Domain Visualization

### Columns = Values

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Domain Visualization

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Cells = State
Alldifferent Constraint

\texttt{alldifferent(L)},

- Built-in of \texttt{ic} library
- No initial propagation possible
- \textit{Suspends}, waits until variables are changed
- When variable is fixed, remove value from domain of other variables
- \textit{Forward checking}
Alldifferent Visualization

Uses the same representation as the domain visualizer

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Disequality Constraints

\[ S \neq 0, \quad M \neq 0, \]

Remove value from domain

\[ S \in \{1..9\}, \quad M \in \{1..9\} \]

Constraints solved, can be removed
## Domains after Disequality

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</table>
Equality Constraint

- Normalization of linear terms
  - Single occurrence of variable
  - Positive coefficients
- Propagation
Normalization

\[
\begin{align*}
1000 \cdot S + & 100 \cdot E + 10 \cdot N + D \\
+1000 \cdot M + & 100 \cdot O + 10 \cdot R + E \\
10000 \cdot M + & 1000 \cdot O + 100 \cdot N + 10 \cdot E + Y
\end{align*}
\]
Normalization

\[
\begin{align*}
1000*S+ & \quad 100*E+ \quad 10*N+ \quad D \\
+1000*M+ & \quad 100*O+ \quad 10*R+ \quad E \\
10000*M+ & \quad 1000*O+ \quad 100*N+ \quad 10*E+ \quad Y
\end{align*}
\]
Normalization

\[
\begin{align*}
1000* S &+ 100* E &+ 10* N &+ D \\
+ 100* O &+ 10* R &+ E \\
\hline
9000* M &+ 1000* O &+ 100* N &+ 10* E &+ Y
\end{align*}
\]
Normalization

\[
\begin{align*}
1000S + & \quad 100E + \quad 10N + \quad D \\
+ & \quad 100O + \quad 10R + \quad E \\
9000M + & \quad 1000O + \quad 100N + \quad 10E + \quad Y
\end{align*}
\]
Normalization

\[
\begin{align*}
&1000*S + 100*E + 10*N + D \\
&\quad + 10*R + E \\
\hline
&9000*M + 900*O + 100*N + 10*E + Y
\end{align*}
\]
Normalization

\[
\begin{align*}
1000*S + & \quad 100*E + \quad 10*N + \quad D \\
+ & \quad 10*R + \quad E \\
9000*M + & \quad 900*O + \quad 100*N + \quad 10*E + \quad Y
\end{align*}
\]
Normalization

\[
\begin{align*}
1000 \cdot S + 100 \cdot E + & \quad D \\
+ 10 \cdot R + & \quad E \\
9000 \cdot M + 900 \cdot O + & \quad 90 \cdot N + 10 \cdot E + & \quad Y
\end{align*}
\]
Normalization

\[
\begin{align*}
1000S &+ 100E &+ D \\
&+ 10R &+ E \\
9000M &+ 900O &+ 90N &+ 10E &+ Y
\end{align*}
\]
Normalization

\[
\begin{align*}
1000*S + 91*E + D \\
+ 10*R \\
9000*M + 900*O + 90*N + Y
\end{align*}
\]
Simplified Equation

\[1000 \times S + 91 \times E + 10 \times R + D = 9000 \times M + 900 \times O + 90 \times N + Y\]
Propagation

\[1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9} =
9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9}\]
Propagation

\[
\begin{align*}
1000 \times S^{1\ldots9} &+ 91 \times E^{0\ldots9} + 10 \times R^{0\ldots9} + D^{0\ldots9} = \\
&1000\ldots9918 \\
9000 \times M^{1\ldots9} &+ 900 \times O^{0\ldots9} + 90 \times N^{0\ldots9} + Y^{0\ldots9} \\
&9000\ldots89919
\end{align*}
\]
Propagation

\[
\begin{align*}
1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9} &= 9000..9918 \\
9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9} &= 9000..9918
\end{align*}
\]
Propagation

\[ 1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9} = \]
\[ 9000..9918 \]
\[ 9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9} \]

Deduction:

\[ M = 1, \ S = 9, \ O \in \{0..1\} \]
Problem
Program
Constraint Setup
Search
Lessons Learned

Domain Definition
Alldifferent Constraint
Disequality Constraints
Equality Constraint

Propagation

\[ \begin{align*} 1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9} &= 9000..9918 \\ 9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9} &= 9000..9918 \end{align*} \]

Deduction:

\[ M = 1, \ S = 9, \ O \in \{0..1\} \]

Why? [Skip]
Consider lower bound for $S$

\[
\underbrace{1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9}}_{9000..9918}
\]

- Lower bound of equation is 9000
- Rest of lhs (left hand side) \((91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9})\) is atmost 918
- $S$ must be greater or equal to \(\frac{9000 - 918}{1000} = 8.082\)
  - otherwise lower bound of equation not reached by lhs
- $S$ is integer, therefore \(S \geq \left\lceil \frac{9000 - 918}{1000} \right\rceil = 9\)
- $S$ has upper bound of 9, so $S = 9$
Consider upper bound of $M$

\[
\frac{1000 \times S^{1..9} + 91 \times E^{0..9} + 10 \times R^{0..9} + D^{0..9}}{9000..9918} = \frac{9000 \times M^{1..9} + 900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9}}{9000..9918}
\]

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $900 \times O^{0..9} + 90 \times N^{0..9} + Y^{0..9}$ is at least 0
- $M$ must be smaller or equal to $\frac{9918 - 0}{9000} = 1.102$
- $M$ must be integer, therefore $M \leq \lfloor \frac{9918 - 0}{9000} \rfloor = 1$
- $M$ has lower bound of 1, so $M = 1$
Consider upper bound of $O$

\[
1000 \star S^{1..9} + 91 \star E^{0..9} + 10 \star R^{0..9} + D^{0..9} = 9000 \star M^{1..9} + 900 \star O^{0..9} + 90 \star N^{0..9} + Y^{0..9}
\]

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $9000 \star 1 + 90 \star N^{0..9} + Y^{0..9}$ is at least 9000
- $O$ must be smaller or equal to $\frac{9918 - 9000}{900} = 1.02$
- $O$ must be integer, therefore $O \leq \left\lfloor \frac{9918 - 9000}{900} \right\rfloor = 1$
- $O$ has lower bound of 0, so $O \in \{0..1\}$
## Propagation of equality: Result

$$
\begin{array}{c|cccccccccc}
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
E & & & & & & & & & & \\
N & & & & & & & & & & \\
D & & & & & & & & & & \\
M & & & & & & & & & & \\
O & & & & & & & & & & \\
R & & & & & & & & & & \\
Y & & & & & & & & & & \\
\end{array}
$$

The result shows the propagation of equality constraints in the problem. The symbols (`*`, `x`) represent the constraints applied and the solution found. The red stars indicate the final solution values.
Problem
Program
Constraint Setup
Search
Lessons Learned

Domain Definition
Alldifferent Constraint
Disequality Constraints
Equality Constraint

Propagation of alldifferent

0
S
E
N
D
M
O
R
Y

1
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6
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7
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8
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9
Y

Y

6

6

6

6

6

6

6

6

Helmut Simonis

Basic Constraint Reasoning

61


## Propagation of alldifferent

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The above grid illustrates a typical problem setup for an alldifferent constraint, where the goal is to ensure that each letter in the grid represents a unique digit from 0 to 9, adhering to specific constraints indicated by the starred cells.
## Propagation of alldifferent

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</table>

The puzzle involves the words "SEND MONEY." After assigning the numbers 4 for the digit 'M,' the propagation of the alldifferent constraint affects the remaining columns.
Propagation of alldifferent

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<table>
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```

S
E
N
D
M
O
R
Y

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## Propagation of alldifferent

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Helmut Simonis

Basic Constraint Reasoning
### Propagation of alldifferent

#### Problem

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\[ O = 0, [E, R, D, N, Y] \in \{2..8\} \]
Waking the equality constraint

- Triggered by assignment of variables
- *or* update of lower or upper bound
Removal of constants

\[ 1000 \times 9 + 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = \\
9000 \times 1 + 900 \times 0 + 90 \times N^{2..8} + Y^{2..8} \]
Removal of constants

\[
1000 \times 9 + 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = \\
9000 \times 1 + 900 \times 0 + 90 \times N^{2..8} + Y^{2..8}
\]
Removal of constants

\[ 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{2..8} + Y^{2..8} \]
Propagation of equality (Iteration 1)

\[
91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{2..8} + Y^{2..8}
\]

\[
\begin{align*}
\underline{204..816} \\
\underline{182..728}
\end{align*}
\]
Propagation of equality (Iteration 1)

\[ 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{2..8} + Y^{2..8} \]

\[ 204..728 \]
Propagation of equality (Iteration 1)

\[
91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{2..8} + Y^{2..8}
\]

\[
204..728
\]

\[
N \geq 3 = \left\lfloor \frac{204 - 8}{90} \right\rfloor, \quad E \leq 7 = \left\lceil \frac{728 - 22}{91} \right\rceil
\]
Propagation of equality (Iteration 2)

\[ 91 \times E^2..7 + 10 \times R^2..8 + D^2..8 = 90 \times N^3..8 + Y^2..8 \]
Propagation of equality (Iteration 2)

\[
91 \times E^{2..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{3..8} + Y^{2..8}
\]

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Basic Constraint Reasoning 75
Propagation of equality (Iteration 2)

\[
91 \times E_{2..7} + 10 \times R_{2..8} + D_{2..8} = 90 \times N_{3..8} + Y_{2..8}
\]

\[
272..725
\]
Propagation of equality (Iteration 2)

\[91 \times E_{2..7} + 10 \times R_{2..8} + D_{2..8} = 90 \times N_{3..8} + Y_{2..8}\]

\[272..725\]

\[E \geq 3 = \left\lfloor \frac{272 - 88}{91} \right\rfloor\]
Propagation of equality (Iteration 3)

\[ 91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{3..8} + Y^{2..8} \]
Propagation of equality (Iteration 3)

\[ 91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{3..8} + Y^{2..8} \]

\[ 295..725 \]

\[ 272..728 \]
Propagation of equality (Iteration 3)

\[
91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{3..8} + Y^{2..8}
\]

\[
\text{295..725}
\]
Propagation of equality (Iteration 3)

\[
91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8} = 90 \cdot N^{3..8} + Y^{2..8}
\]

\[
295..725
\]

\[
N \geq 4 = \left\lfloor \frac{295 - 8}{90} \right\rfloor
\]
Propagation of equality (Iteration 4)

\[ 91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{4..8} + Y^{2..8} \]
Propagation of equality (Iteration 4)

\[ 91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{4..8} + Y^{2..8} \]
Propagation of equality (Iteration 4)

\[
\begin{align*}
91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} &= 90 \times N^{4..8} + Y^{2..8} \\
362..725
\end{align*}
\]
Propagation of equality (Iteration 4)

\[
91 \times E^{3..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{4..8} + Y^{2..8}
\]

\[
\text{362..725}
\]

\[
E \geq 4 = \left\lfloor \frac{362 - 88}{91} \right\rfloor
\]
Propagation of equality (Iteration 5)

\[ 91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{4..8} + Y^{2..8} \]
Propagation of equality (Iteration 5)

\[91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{4..8} + Y^{2..8}\]

\[386..725 + 362..728\]
Propagation of equality (Iteration 5)

\[
91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{4..8} + Y^{2..8} \\
386..725
\]
Propagation of equality (Iteration 5)

\[
91 \cdot E^{4..7} + 10 \cdot R^{2..8} + D^{2..8} = 90 \cdot N^{4..8} + Y^{2..8}
\]

\[
386..725
\]

\[
N \geq 5 = \left\lceil \frac{386 - 8}{90} \right\rceil
\]
Propagation of equality (Iteration 6)

\[ 91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8} \]
Propagation of equality (Iteration 6)

\[
91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8}
\]

\[
\begin{align*}
386..725 & \quad 452..728
\end{align*}
\]
Propagation of equality (Iteration 6)

\[ 91 \cdot E^{4..7} + 10 \cdot R^{2..8} + D^{2..8} = 90 \cdot N^{5..8} + Y^{2..8} \]

\[ 452..725 \]
Propagation of equality (Iteration 6)

\[
91 \cdot E^{4..7} + 10 \cdot R^{2..8} + D^{2..8} = 90 \cdot N^{5..8} + Y^{2..8} \\
= \frac{452 - 8}{90}
\]

\[
N \geq 5 = \left\lceil \frac{452 - 8}{90} \right\rceil, \quad E \geq 4 = \left\lceil \frac{452 - 88}{91} \right\rceil
\]

No further propagation at this point
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Domains after setup
**labeling built-in**

`labeling([S,E,N,D,M,O,R,Y])`

- Try variable is order given
- Try values starting from smallest value in domain
- When failing, backtrack to last open choice
- *Chronological Backtracking*
- *Depth First search*
Variable S already fixed
Step 2, Alternative $E = 4$

Variable $E \in \{4..7\}$, first value tested is 4
Assignment $E = 4$
Propagation of $E = 4$, equality constraint

$$91 \times 4 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8}$$
Propagation of $E = 4$, equality constraint

\[
\begin{align*}
91 \times 4 &+ 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8} \\
386..452 &+ 452..728
\end{align*}
\]
Propagation of $E = 4$, equality constraint

\[
91 \times 4 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8}
\]

452
Propagation of $E = 4$, equality constraint

$$\begin{align*}
91 \times 4 + 10 \times R^{2..8} + D^{2..8} &= 90 \times N^{5..8} + Y^{2..8} \\
452 &= 452
\end{align*}$$

$N = 5, Y = 2, R = 8, D = 8$
## Result of equality propagation

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**Helmut Simonis**

*Basic Constraint Reasoning*
### Propagation of alldifferent

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### Notes
- **Red stars** indicate propagation due to alldifferent constraint.
- The constraint states that all variables must have different values.
Propagating alldifferent

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Alldifferent fails!
Step 2, Alternative $E = 5$

Return to last open choice, $E$, and test next value
**Assignment** \( E = 5 \)

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Helmut Simonis  Basic Constraint Reasoning  109
### Propagation of alldifferent

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**Solution**

- **SENDMORY**
- **0123456789**

**Lessons Learned**

- Propagation of alldifferent constraints can be effective in solving complex problems.
- Understanding the propagation rules is crucial for efficient problem solving.

**Further Steps**

- Explore alternative constraint propagation strategies.
- Test different constraint programming techniques.

**Basic Constraint Reasoning 110**
Propagation of alldifferent

\[ N \neq 5, \quad N \geq 6 \]
Propagation of equality

\[ 91 \times 5 + 10 \times R_{2..8} + D_{2..8} = 90 \times N_{6..8} + Y_{2..8} \]
Propagation of equality

91 \times 5 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{6..8} + Y^{2..8}

\begin{align*}
477..543 & + 542..728 \\
\end{align*}
Propagation of equality

\[ 91 \times 5 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{6..8} + Y^{2..8} \]

\[ 542..543 \]
Propagation of equality

\[\begin{align*}
91 \times 5 + 10 \times R^{2..8} + D^{2..8} &= 90 \times N^{6..8} + Y^{2..8} \\
542..543
\end{align*}\]

\[N = 6, \ Y \in \{2, 3\}, \ R = 8, \ D \in \{7..8\}\]
Result of equality propagation

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- S: German for 'Sun'
- E: German for 'Empire'
- N: German for 'Note'
- D: German for 'Door'
- M: German for 'Mord'
- O: German for 'On'
- R: German for 'Richtung'
- Y: German for 'Year'

- Starred cells indicate a fixed value.
- Crossed cells indicate an equality constraint.
## Propagation of `alldifferent`

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- **Step 1**: Initial configuration
- **Step 2**: Propagation of constraints
- **Further Steps**: Further propagation and solving

*Helmut Simonis: Basic Constraint Reasoning*
Propagation of \textit{alldifferent}

\[
\begin{array}{cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
S &   &   &   &   &   &   &   &   &   & 7 \\
E &   &   &   &   &   &   & 5 &   &   &   \\
N &   &   &   &   &   &   & 3 &   &   &   \\
D &   &   &   &   &   & 7 &   &   &   &   \\
M &   & 7 &   &   &   &   &   &   &   &   \\
O & 7 &   &   &   &   &   &   &   &   &   \\
R &   &   &   &   &   &   &   &   & 9 &   \\
Y &   &   &   &   &   &   &   &   &   &   \\
\end{array}
\]

\[ D = 7 \]
Propagation of equality

\[91 \times 5 + 10 \times 8 + 7 = 90 \times 6 + Y^2 \ldots^3\]
Propagation of equality

\[ 91 \times 5 + 10 \times 8 + 7 = 90 \times 6 + Y^{2..3} \]

\[ 542 = 542..543 \]
Propagation of equality

\[ 91 \times 5 + 10 \times 8 + 7 = 90 \times 6 + Y^{2..3} \]

542
Propagation of equality

\[ 91 \cdot 5 + 10 \cdot 8 + 7 = 90 \cdot 6 + Y^{2..3} \]

\[ 542 \]

\[ Y = 2 \]
Last propagation step

```
  0 1 2 3 4 5 6 7 8 9
S
E
N
D
M
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-\star
```

S: SEND
E: END
D: EDMOR
M: MOR
O: OR
Y: Y
Further Steps: Nothing more to do
Further Steps: Nothing more to do
Further Steps: Nothing more to do
Further Steps: Nothing more to do
Further Steps: Nothing more to do
Further Steps: Nothing more to do
Further Steps: Nothing more to do
Solution

\[
\begin{array}{cccc}
9 & 5 & 6 & 7 \\
+ & 1 & 0 & 8 & 5 \\
\hline
1 & 0 & 6 & 5 & 2
\end{array}
\]
Outline

1. Problem
2. Program
3. Constraint Setup
4. Search
5. Lessons Learned
Topics introduced

- Finite Domain Solver in ECLiPSe, ic library
- Models and Programs
- Constraint Propagation and Search
- Basic constraints: linear arithmetic, alldifferent, disequality
- Built-in search: labeling
- Visualizers for variables, constraints and search
Lessons Learned

- Constraint models are expressed by variables and constraints.
- Problems can have many different models, which can behave quite differently. Choosing the best model is an art.
- Constraints can take many different forms.
- Propagation deals with the interaction of variables and constraints.
- It removes some values that are inconsistent with a constraint from the domain of a variable.
- Constraints only communicate via shared variables.
Lessons Learned

- Propagation usually is not sufficient, search may be required to find a solution.
- Propagation is data driven, and can be quite complex even for small examples.
- The default search uses chronological depth-first backtracking, systematically exploring the complete search space.
- The search choices and propagation are interleaved, after every choice some more propagation may further reduce the problem.
Outline

6 Alternative Models
   - Model without Disequality
   - Multiple Equations

7 Exercises
Alternative 1

- Do we need the constraint “Numbers do not begin with a zero”?
- This is not given explicitly in the problem statement
- Remove disequality constraints from program
- Previous solution is still a solution
- Does it change propagation?
- Does it have more solutions?
Listing 1: Alternative 1

:- module(alternative1).
:- export(sendmory/1).
:- lib(ic).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
### After Setup without Disequality

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The grid represents the constraints and values after setup without disequality, with some cells filled in灰色. This is likely part of a constraint satisfaction problem, where the goal is to assign values to the cells such that certain constraints are satisfied.
Setup Comparison

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Search Tree: Many Solutions
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Search Tree: Many Solutions
Search Tree: Many Solutions

The diagram shows a search tree with multiple solutions. The tree branches are labeled with nodes 0, 1, and 2, leading to a final solution labeled 'E'.
Search Tree: Many Solutions

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Search Tree: Many Solutions
Search Tree: Many Solutions
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Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions

Helmut Simonis Basic Constraint Reasoning 156
Search Tree: Many Solutions
Search Tree: Many Solutions

Alternative Models
Exercises
Model without Disequality
Multiple Equations
Search Tree: Many Solutions
Search Tree: Many Solutions

Alternative Models
Exercises

Model without Disequality
Multiple Equations
Search Tree: Many Solutions
Search Tree: Many Solutions

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Search Tree: Many Solutions
Search Tree: Many Solutions

The search tree displays multiple solutions for the constraint problem. Each node represents a decision point, and the branches show the different paths taken to explore possible solutions. The tree illustrates how the algorithm navigates through the constraints to find valid solutions.
Search Tree: Many Solutions

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Search Tree: Many Solutions
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Search Tree: Many Solutions

Back to Start  Skip Animation

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Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions

- **Model without Disequality**
- **Multiple Equations**

Diagram of a search tree with labels and values.
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions
Search Tree: Many Solutions

Alternative Models
Exercises
Model without Disequality
Multiple Equations

Back to Start  Skip Animation
Search Tree: Many Solutions
Search Tree: Many Solutions
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Helmut Simonis
Basic Constraint Reasoning

Back to Start  Skip Animation
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Search Tree: Many Solutions
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Not just a different model, solving a different problem!
Often we can choose which problem we want to solve
   Which constraints to include
   What to ignore
In this case not acceptable
Alternative 2

- Large equality difficult to understand by humans
- Replace with multiple, simpler equations
- Linked by carry variables (0/1)
- Should produce same solutions
- Does it give same propagation?

\[
\begin{array}{cccccc}
S & E & N & D \\
+ & M & O & R & E \\
\hline
+ C_5 & C_4 & C_3 & C_2 \\
M & O & N & E & Y
\end{array}
\]
Carry Variables with Multiple Equations

:-module(alternative2), export(sendmory/1), lib(ic).
sendmory(L):-
    L=[S,E,N,D,M,O,R,Y], L :: 0..9, [C2,C3,C4,C5] :: 0..1, alldifferent(L), S #\= 0, M #\= 0, M #= C5, S+M+C4 #= 10*C5+O, E+O+C3 #= 10*C4+N, N+R+C2 #= 10*C3+E, D+E #= 10*C2+Y, labeling(L).
Carry Variables with Multiple Equations

:-module(alternative2),export(sendmory/1),lib(ic).

sendmory(L):=-▷ same as before
    L=[S,E,N,D,M,O,R,Y], L :: 0..9,
    [C2,C3,C4,C5] :: 0..1,
    alldifferent(L),
    S \= 0, M \= 0,
    M = C5,
    S+M+C4 \= 10*C5+O,
    E+O+C3 \= 10*C4+N,
    N+R+C2 \= 10*C3+E,
    D+E \= 10*C2+Y,
    labeling(L).
Carry Variables with Multiple Equations

```
:-module(alternative2), export(sendmory/1), lib(ic).
sendmory(L):-
    L=[S,E,N,D,M,O,R,Y], L :: 0..9,
    [C2,C3,C4,C5] :: 0..1, new
    alldifferent(L),
    S #\= 0, M #\= 0,
    M #= C5,
    S+M+C4 #= 10*C5+O,
    E+O+C3 #= 10*C4+N,
    N+R+C2 #= 10*C3+E,
    D+E #= 10*C2+Y,
    labeling(L).
```
Carry Variables with Multiple Equations

:-module(alternative2),export(sendmory/1),lib(ic).
sendmory(L):-
    L=[S,E,N,D,M,O,R,Y],L :: 0..9,
    [C2,C3,C4,C5] :: 0..1,
    alldifferent(L),
    S #\= 0,M #\= 0,
    M #= C5,
    S+M+C4 #= 10*C5+O,
    E+O+C3 #= 10*C4+N,
    N+R+C2 #= 10*C3+E,
    D+E     #= 10*C2+Y,
    labeling(L).
With Carry Variables: After Setup

```
    0  1  2  3  4  5  6  7  8  9
  S  
  E  
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O  
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 Y
```

Helmut Simonis
Basic Constraint Reasoning
### Setup Comparison

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Comparison

Single Equation

Multiple Equations
Observations

- This is solving the original problem
- Search tree slightly bigger
- Caused here by missing interaction of equations
- And repeated variables
- But: Introducing auxiliary variables not always bad!
Henry Dudeney.
Send+More=Money.
*Strand Magazine*, Volume 68:pages 97 and 214, July 1924.

Henry Dudeney.
*Amusements in Mathematics.*
Project Gutenberg, 1917.
http://www.gutenberg.org/etext/16713.
Outline

6 Alternative Models

7 Exercises
Exercises

1. Does the reasoning for the equality constraints that we have presented remove all inconsistent values? Consider the constraint $Y=2*X$.

2. Why is it important to remove multiple occurrences of the same variable from an equality constraint? Give an example!

3. Solve the puzzle DONALD+GERALD=ROBERT. What is the state of the variables before the search, after the initial constraint propagation?

4. Solve the puzzle $Y*WORRY = DOOOOD$. What is different?

5. (extra credit) How would you design a program that finds new crypt-arithmetic puzzles? What makes a good puzzle?