Chapter 4: Basic Constraint Reasoning
(SEND+MORE=MONEY)

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ECLiPSe ELearning

Overview

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Outline

1. Problem
2. Program
3. Constraint Setup
4. Search
5. Lessons Learned

What we want to introduce

- Finite Domain Solver in ECLiPSe
- Models and Programs
- Constraint Propagation and Search
- Basic constraints: linear arithmetic, alldifferent, disequality
- Built-in search: Labeling
- Visualizers for variables, constraints and search
A Crypt-Arithmetic Puzzle

We begin with the definition of the SEND + MORE = MONEY puzzle. It is often shown in the form of a hand-written addition:

\[
\begin{array}{c}
S E N D \\
+ M O R E \\
\hline
M O N E Y \\
\end{array}
\]

Rules

- Each character stands for a digit from 0 to 9.
- Numbers are built from digits in the usual, positional notation.
- Repeated occurrence of the same character denote the same digit.
- Different characters denote different digits.
- Numbers do not start with a zero.
- The equation must hold.
Each character is a variable, which ranges over the values 0 to 9.

An *alldifferent* constraint between all variables, which states that two different variables must have different values. This is a very common constraint, which we will encounter in many other problems later on.

Two *disequality constraints* (variable $X$ must be different from value $V$) stating that the variables at the beginning of a number cannot take the value 0.

An arithmetic *equality constraint* linking all variables with the proper coefficients and stating that the equation must hold.

```prolog
:- module(sendmory).
:- export(sendmory/1).
:- lib(ic).
sendmory(L):-
    L = [S,E,N,D,M,O,R,Y], % Variables
    L :: 0..9,
    alldifferent(L), % Constraints
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L). % Search
```
Choice of Model

- This is *one* model, not *the* model of the problem
- Many possible alternatives
- Choice often depends on your constraint system
  - Constraints available
  - Reasoning attached to constraints
- Not always clear which is the *best* model
- Often: Not clear what is the *problem*

Running the program

- To run the program, we have to enter the query
  - `sendmory:sendmory(L).`
- Result
  - `L = [9, 5, 6, 7, 1, 0, 8, 2]`
  - `yes (0.00s cpu, solution 1, maybe more)`
But how did the program come up with this solution?

\[ L = [S, E, N, D, M, O, R, Y], \]
\[ L :: 0..9, \]
\[ [S, E, N, D, M, O, R, Y] \in \{0..9\} \]
Domain Visualization

Columns = Values

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Rows = Variables

Cells = State

Alldifferent Constraint

\[ \text{alldifferent}(L), \]

- Built-in of \text{ic} library
- No initial propagation possible
- \text{Suspends}, waits until variables are changed
- When variable is fixed, remove value from domain of other variables
- \text{Forward checking}
Alldifferent Visualization

Uses the same representation as the domain visualizer

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Disequality Constraints

\[ S \neq 0, M \neq 0, \]

Remove value from domain

\[ S \in \{1..9\}, M \in \{1..9\} \]

Constraints solved, can be removed
Domains after Disequality

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Normalization of linear terms
- Single occurrence of variable
- Positive coefficients

Propagation
Normalization

\[
\begin{align*}
1000S &  
100E & 
10N & 
D \\
+1000M & 
100O & 
10R & 
E \\
10000M & 
1000O & 
100N & 
10E & 
Y \\
\end{align*}
\]

is transformed into

\[
\begin{align*}
1000S & 
91E & 
10R & 
D \\
+ & 
10E \\
9000M & 
900O & 
90N & 
Y \\
\end{align*}
\]

Simplified Equation

\[
1000 \times S + 91 \times E + 10 \times R + D = 9000 \times M + 900 \times O + 90 \times N + Y
\]
Consider lower bound for $S$

\[
\begin{align*}
1000 \cdot S^{1..9} + 91 \cdot E^{0..9} + 10 \cdot R^{0..9} + D^{0..9} &= 9000 \cdot M^{1..9} + 900 \cdot O^{0..9} + 90 \cdot N^{0..9} + Y^{0..9} \\
\quad &\quad 9000..9918
\end{align*}
\]

- Lower bound of equation is 9000
- Rest of Lhs (left hand side) $(91 \cdot E^{0..9} + 10 \cdot R^{0..9} + D^{0..9})$ is atmost 918
- $S$ must be greater or equal to $\frac{9000-918}{1000} = 8.082$
  - otherwise lower bound of equation not reached by Lhs
- $S$ is integer, therefore $S \geq \lceil \frac{9000-918}{1000} \rceil = 9$
- $S$ has upper bound of 9, so $S = 9$
Consider upper bound of \( M \)

\[
\frac{1000 \times S^{1.9} + 91 \times E^{0.9} + 10 \times R^{0.9} + D^{0.9}}{9000..9918} = \frac{9000 \times M^{1.9} + 900 \times O^{0.9} + 90 \times N^{0.9} + Y^{0.9}}{9000..9918}
\]

- Upper bound of equation is 9918
- Rest of rhs (right hand side) \( 900 \times O^{0.9} + 90 \times N^{0.9} + Y^{0.9} \) is at least 0
- \( M \) must be smaller or equal to \( \frac{9918 - 0}{9000} = 1.102 \)
- \( M \) must be integer, therefore \( M \leq \lfloor \frac{9918 - 0}{9000} \rfloor = 1 \)
- \( M \) has lower bound of 1, so \( M = 1 \)

Consider upper bound of \( O \)

\[
\frac{1000 \times S^{1.9} + 91 \times E^{0.9} + 10 \times R^{0.9} + D^{0.9}}{9000..9918} = \frac{9000 \times M^{1.9} + 900 \times O^{0.9} + 90 \times N^{0.9} + Y^{0.9}}{9000..9918}
\]

- Upper bound of equation is 9918
- Rest of rhs (right hand side) \( 9000 \times 1 + 90 \times N^{0.9} + Y^{0.9} \) is at least 9000
- \( O \) must be smaller or equal to \( \frac{9918 - 9000}{900} = 1.02 \)
- \( O \) must be integer, therefore \( O \leq \lfloor \frac{9918 - 9000}{900} \rfloor = 1 \)
- \( O \) has lower bound of 0, so \( O \in \{0..1\} \)
### Propagation of equality: Result

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### Propagation of alldifferent

\[
O = 0, [E, R, D, N, Y] \in \{2..8\}
\]
Waking the equality constraint

- Triggered by assignment of variables
- or update of lower or upper bound

Removal of constants

\[ 1000 \times 9 + 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = \\
9000 \times 1 + 900 \times 0 + 90 \times N^{2..8} + Y^{2..8} \]

\[ 1000 \times 9 + 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = \\
9000 \times 1 + 900 \times 0 + 90 \times N^{2..8} + Y^{2..8} \]

\[ 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{2..8} + Y^{2..8} \]
Propagation of equality (Iteration 1)

\[ 91 \times E^{2..8} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{2..8} + Y^{2..8} \]
\[ 204.816 \]
\[ 182.728 \]

\[ N \geq 3 = \left\lceil \frac{204 - 8}{90} \right\rceil, \quad E \leq 7 = \left\lfloor \frac{728 - 22}{91} \right\rfloor \]

Propagation of equality (Iteration 2)

\[ 91 \times E^{2..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{3..8} + Y^{2..8} \]
\[ 204.725 \]
\[ 272.728 \]

\[ E \geq 3 = \left\lceil \frac{272 - 88}{91} \right\rceil \]
Propagation of equality (Iteration 3)

\[ 91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8} = 90 \cdot N^{3..8} + Y^{2..8} \]

\[ \frac{91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8}}{295..725} = \frac{90 \cdot N^{3..8} + Y^{2..8}}{272..728} \]

\[ \frac{91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8}}{295..725} = 90 \cdot N^{3..8} + Y^{2..8} \]

\[ N \geq 4 = \left \lfloor \frac{295 - 8}{90} \right \rfloor \]

Propagation of equality (Iteration 4)

\[ 91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8} = 90 \cdot N^{4..8} + Y^{2..8} \]

\[ \frac{91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8}}{295..725} = \frac{90 \cdot N^{4..8} + Y^{2..8}}{362..728} \]

\[ \frac{91 \cdot E^{3..7} + 10 \cdot R^{2..8} + D^{2..8}}{362..725} = 90 \cdot N^{4..8} + Y^{2..8} \]

\[ E \geq 4 = \left \lfloor \frac{362 - 88}{91} \right \rfloor \]
Propagation of equality (Iteration 5)

\[ 91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{4..8} + Y^{2..8} \]

\[ 91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = \frac{386.725}{362.728} \times 90 \times N^{4..8} + Y^{2..8} \]

\[ 91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = \frac{386.725}{386.725} \times 90 \times N^{4..8} + Y^{2..8} \]

\[ N \geq 5 = \left\lceil \frac{386 - 8}{90} \right\rceil \]

Propagation of equality (Iteration 6)

\[ 91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8} \]

\[ 91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = \frac{386.725}{452.728} \times 90 \times N^{5..8} + Y^{2..8} \]

\[ 91 \times E^{4..7} + 10 \times R^{2..8} + D^{2..8} = \frac{452.725}{452.725} \times 90 \times N^{5..8} + Y^{2..8} \]

\[ N \geq 5 = \left\lceil \frac{452 - 8}{90} \right\rceil, E \geq 4 = \left\lceil \frac{452 - 88}{91} \right\rceil \]

No further propagation at this point
labeling built-in

labeling([S, E, N, D, M, O, R, Y])

- Try variable is order given
- Try values starting from smallest value in domain
- When failing, backtrack to last open choice
- *Chronological Backtracking*
- *Depth First search*
Variable $S$ already fixed

Step 2, Alternative $E = 4$

Variable $E \in \{4..7\}$, first value tested is 4
Assignment $E = 4$

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Propagation of $E = 4$, equality constraint

\[ 91 \times 4 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8} \]

\[ 91 \times 4 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8} \]

\[ 91 \times 4 + 10 \times R^{2..8} + D^{2..8} = 90 \times N^{5..8} + Y^{2..8} \]

\[ N = 5, \; Y = 2, \; R = 8, \; D = 8 \]
Result of equality propagation

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Alldifferent fails!
Step 2, Alternative $E = 5$

Return to last open choice, $E$, and test next value

Assignment $E = 5$
Propagation of alldifferent

\[ N \neq 5, \quad N \geq 6 \]

\[ 91 \times 5 + 10 \times R^2.8 + D^2.8 = 90 \times N^6.8 + Y^{2.8} \]

\[ 477.543 \]

\[ 542.728 \]

\[ 91 \times 5 + 10 \times R^2.8 + D^2.8 = 90 \times N^6.8 + Y^{2.8} \]

\[ 542.543 \]

\[ N = 6, \quad Y \in \{2, 3\}, \quad R = 8, \quad D \in \{7, 8\} \]
Result of equality propagation

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$D = 7$

Propagation of \textit{alldifferent}
Propagation of equality

\[ 91 \times 5 + 10 \times 8 + 7 = 90 \times 6 + Y^{2..3} \]

\[ 91 \times 5 + 10 \times 8 + 7 = 90 \times 6 + Y^{2..3} \]
\[ \underline{542} \]
\[ Y = 2 \]

Last propagation step

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Helmut Simonis Basic Constraint Reasoning 49
Complete Search Tree

Solution

\[
\begin{array}{c}
9 & 5 & 6 & 7 \\
+ & 1 & 0 & 8 & 5 \\
\hline
1 & 0 & 6 & 5 & 2
\end{array}
\]
Topics introduced

- Finite Domain Solver in ECLiPSe, `ic library`
- Models and Programs
- Constraint Propagation and Search
- Basic constraints: linear arithmetic, `alldifferent`, disequality
- Built-in search: `labeling`
- Visualizers for variables, constraints and search

Lessons Learned

- Constraint models are expressed by variables and constraints.
- Problems can have many different models, which can behave quite differently. Choosing the best model is an art.
- Constraints can take many different forms.
- Propagation deals with the interaction of variables and constraints.
- It removes some values that are inconsistent with a constraint from the domain of a variable.
- Constraints only communicate via shared variables.
Lessons Learned

- Propagation usually is not sufficient, search may be required to find a solution.
- Propagation is data driven, and can be quite complex even for small examples.
- The default search uses chronological depth-first backtracking, systematically exploring the complete search space.
- The search choices and propagation are interleaved, after every choice some more propagation may further reduce the problem.

Alternative 1

- Do we need the constraint “Numbers do not begin with a zero”?
- This is not given explicitly in the problem statement
- Remove disequality constraints from program
- Previous solution is still a solution
- Does it change propagation?
- Does it have more solutions?
Listing 1: Alternative 1

:-module(alternative1).
:-export(sendmory/1).
:-lib(ic).

sendmory(L):-  
    L = [S,E,N,D,M,O,R,Y], 
    L :: 0..9, 
    all diferentes(L), 
    1000*S + 100*E + 10*N + D + 
    1000*M + 100*O + 10*R + E #= 
    10000*M + 1000*O + 100*N + 10*E + Y, 
    labeling(L).

After Setup without Disequality
Setup Comparison

Alternative Models
Exercises
Model without Disequality
Multiple Equations

original

```
  0 1 2 3 4 5 6 7 8 9
 S
 E
 N  D  M
 O
 R
 Y
```

alternative 1

```
  0 1 2 3 4 5 6 7 8 9
 S
 E
 N
 D  M
 O
 R
 Y
```

Search Tree: Many Solutions

```
```

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Note:

- Not just a different model, solving a different problem!
- Often we can choose which problem we want to solve
  - Which constraints to include
  - What to ignore
- In this case not acceptable

Alternative 2

- Large equality difficult to understand by humans
- Replace with multiple, simpler equations
- Linked by carry variables (0/1)
- Should produce same solutions
- Does it give same propagation?

\[
\begin{array}{cccccc}
S & E & N & D \\
+ & M & O & R & E \\
\hline
+ & C_5 & C_4 & C_3 & C_2 \\
M & O & N & E & Y
\end{array}
\]
Carry Variables with Multiple Equations

:-module(alternative2),export(sendmory/1),lib(ic).
sendmory(L):=\same as before
    L=[S,E,N,D,M,O,R,Y],L :: 0..9,
    [C2,C3,C4,C5] :: 0..1, \new
    alldifferent(L),
    S \#= 0,M \#= 0,
    M \= C5,
    S+M+C4 \= 10*C5+O,
    E+O+C3 \= 10*C4+N,
    N+R+C2 \= 10*C3+E,
    D+E \= 10*C2+Y,
    labeling(L).
This is solving the original problem
Search tree slightly bigger
Caused here by missing interaction of equations
And repeated variables
But: Introducing auxiliary variables not always bad!
Exercises

1. Does the reasoning for the equality constraints that we have presented remove all inconsistent values? Consider the constraint Y=2*X.
2. Why is it important to remove multiple occurrences of the same variable from an equality constraint? Give an example!
3. Solve the puzzle DONALD+GERALD=ROBERT. What is the state of the variables before the search, after the initial constraint propagation?
4. Solve the puzzle Y*WORRY = DOOOOD. What is different?
5. (extra credit) How would you design a program that finds new crypt-arithmetic puzzles? What makes a good puzzle?