A Note on Perfect Square Placement

N. Beldiceanu
E. Bourreau
H. Simonis

COSYTEC SA
4, rue Jean Rostand
F-91893 Orsay Cedex
France
{beldiceanu,bourreau,simonis}@cosytec.com

Abstract

In this note we describe the solution of perfect square placement problems with CHIP. A set of 207 perfect square placement problems from [BD92] is used. We present the constraint model in CHIP, which uses the global constraints diffn and cumulative together with a specific labeling routine for perfect placement problems. We show that all problems can be solved and present backtracking count and execution times for finding the first solution and for exploring the complete search space.

Problem

We discuss the following perfect square placement problem (also called squared square problem [CFG91][LW92]):

Pack a set of squares with given integer sizes into a bigger square in such a way that no squares overlap each other and all square borders are parallel to the border of the big square. For a perfect placement problem, all squares have different sizes. The sum of the square surfaces is equal to the surface of the packing square, so that there is no spare capacity. The problem data come from [BD92], which lists all simple perfect square placement problems of order 21 to 25. Simple in this context means that the problems do not contain a subset of squares (greater than one) which are placed in a rectangle.
Duijvestijn has shown in 1962 that no instances exist with less than 21 squares [Dui78]. A single problem exists with 21 squares. In this problem, the squares have sizes [2,4,6,7,8,9,11,15,16,17,18,19,24,25,27,29,33,35,37,42,50] and must be packed into a square of size 112. A solution is shown below. There are 8 problems of order 22, 12 of order 23, 26 of order 24, and 160 of order 25. A constraint approach for this problem was already studied in [AB93], where a solution is found in 38 seconds. The same problem was mentioned in [VH94], where a solution was found in 60 seconds. Both results were obtained on SUN workstations. With the program shown here, we find a solution in 0.3 seconds on a PC.

Program

The CHIP program to solve the square placement problem is given below. Each square is defined by two domain variables X and Y and integer Width and Height values. The predicate run has an input parameter, which is the problem instance number and two output parameters, the lists of the X and Y values for the coordinates

of the bottom left corners of the squares. The predicate gen_data/6 reads the problem definition of the instance and generates lists of domain variables for the X and Y values. The square sizes are given in decreasing order, the largest square is the first entry in the lists. The predicate gen_rect/5 creates a list of rectangles (each a list of four variables X, Y, Width and Height) from the four parameter lists.
The diffn [BC94] constraint states that the set of squares do not overlap and fit into the available space. The first argument is a list of rectangles, generated by gen_rect/5. The fourth argument defines the available space.

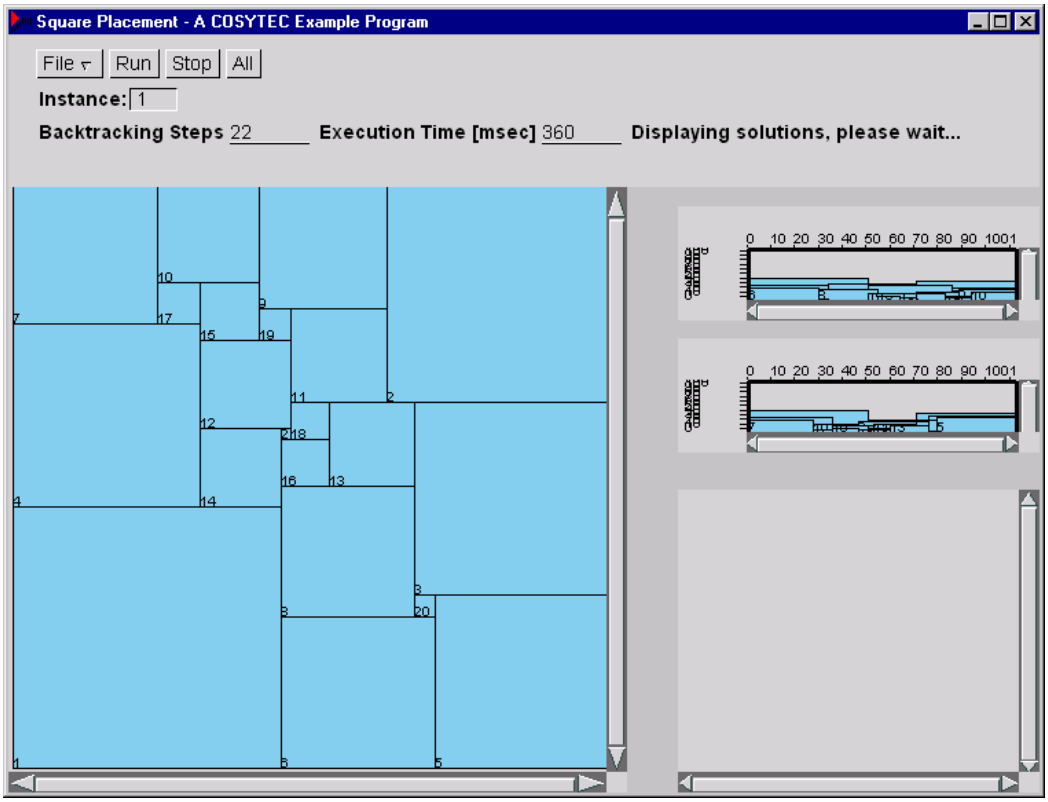

Figure 1: Solution of the 21 square problem

We also use two redundant cumulative [AB93] constraints which project the placement in x or y dimension. The first argument gives the origin, the second argument the size and the third argument the height of the squares to be placed. Arguments 6 and 7 define the available placement space. The argument 8 enforces an additional constraint that the available space must be completely used.
The labeling routine is described in [AB93]. We first find an assignment of all X variables, and then find an assignment for the Y variables. Many other assignment strategies are possible, the one shown here gives the best results on perfect placement problems. The labeling method works by finding the smallest value in the domain of the unassigned variables and then assigning this value to some variable. On backtracking, other variables will be assigned to this value. The predicate lmindomain/2/3 finds the smallest value a list of domain variables. The predicate fix_min/3 is a non-deterministic predicate which either fixes a variable to the minimal value

```
run(Nr,LX,LY):-
      gen_data(Nr,LX,LY,LW,LH,Max),
      gen_rect(LX,LY,LW,LH,Lrect),
      Max1 is Max + 1,
      diffn(Lrect,unused,unused,[Max1,Max1]),
      cumulative(LX,LW,LH,unused,unused,Max,Max1,[Max,0,1]),
      cumulative(LY,LH,LW,unused,unused,Max,Max1,[Max,0,1]),
      labeling(LX),
      labeling(LY).

gen_data(Nr,LX,LY,Sizes,Sizes,MaxPlace) :-
      data(Nr,NbSquare,MaxPlace,SizeSquares),
      length(LX,NbSquare),
      length(LY,NbSquare),
      LX :: 1..MaxPlace,
      LY :: 1..MaxPlace,
      reverse(SizeSquares,Sizes).

gen_rect([],[],[],[],[]).
gen_rect([X|Rx],[Y|Ry],[W|Rw],[H|Rh],[[X,Y,W,H]|R]) :-
      gen_rect(Rx,Ry,Rw,Rh,R).

labeling([]).
labeling([X|Y]) :-
      lmindomain([X|Y], M),
      fix_min([X|Y], M, R),
      labeling(R).

fix_min([V|R], M, R) :-
      V = M,
      inc_choice.
fix_min([V|R], M, [V|S]) :-
      V #> M,
      fix_min(R, M, S).

lmindomain([H|T], M) :-
      domain_info(H, Min, _, _, _, _),
      lmindomain(T, Min, M).

lmindomain([], M, M).
lmindomain([X|Y], M, Mend) :-
      domain_info(X, Min, _, _, _, _),
      M1 is min(M,Min),
      lmindomain(Y, M1, Mend).

inc_choice.
inc_choice:-
      incval(choice,_),
      fail.
```

Program 1 : Square placement program in CHIP

Results

The table below gives the results for CHIP V5.2 on the benchmark set. The results were obtained with the commercial CHIP version on a Pentium MMX 233MHz PC, running WindowsNT 4.0 with 64 Mb of memory. The table shows the instance number, the number of backtracking steps and the execution time required to find a first solution, the total number of solutions found and the backtracking count and execution times to find all solutions by complete enumeration of the search space. Note that for most problems 8 solutions are found, these correspond to one solution under all plane symmetries. Problems 166 and 167 (also 168 and 169, and 182 and 183) have identical data, but two independent (non-isomorphic) solutions. These problems are given in this form in the original data set.

The maximum time to find a first solution was spent on problem 59 with 32.5 seconds (3877 backtracking steps), the maximum time to explore the complete search space was spent in problem 48 with 503 seconds (66100 backtracking steps). On the other hand, for 38 problems a first solution was found without backtracking, and 94 problems were explored completely with less than 2000 backtracking steps.

Instance	Backtrack for first solution	Time for first solution [ms]	Number of solutions	Backtrack for all solutions and proof	Time for all solutions and proof [ms]
1	22	341	8	1139	5608
2	180	1242	8	3644	23604
3	53	511	8	2500	14311
4	28	250	8	1143	5958
5	94	701	8	1273	6228
6	93	661	8	1491	7802
7	11	190	8	891	4316
8	6	160	8	1004	4827
9	0	140	8	1140	5818
10	14	231	8	4265	18747
11	177	1312	8	3874	21782
12	7	120	8	1634	7941
13	5	150	8	2895	16253
14	659	4206	8	2406	14311
15	2795	20509	8	7019	50573
16	818	6199	8	4813	32937
17	40	341	8	1298	5268
18	1	150	8	1046	4486
19	6	190	8	1578	7150
20	239	1503	8	1991	11557
21	9	230	8	781	3125
22	895	4987	8	8396	44594
23	1862	15943	8	6627	49050
24	393	2884	8	4114	24105
25	50	431	8	1610	9293
26	1041	10245	8	3701	29012
27	70	471	8	3117	17575
28	153	1342	8	5080	26599

29	642	5127	8	3987	27489
30	572	4697	8	3066	22082
31	10	240	8	2227	14010
32	12	270	8	3441	23844
33	211	1993	8	3417	18066
34	0	141	8	2379	16224
35	443	3655	8	2803	20319
36	2	180	8	676	3295
37	1	190	8	1312	8272
38	49	470	8	2533	16333
39	9	250	8	1490	9203
40	8	271	8	3117	21471
41	18	311	8	1044	6500
42	28	460	8	1629	10765
43	0	250	8	818	5858
44	0	201	8	1297	9274
45	3	230	8	839	5388
46	4	190	8	1576	10074
47	0	200	8	1286	5648
48	81	891	8	66100	503094
49	1275	10455	8	8760	59705
50	64	722	8	4336	26959
51	98	982	8	1747	10636
52	359	3174	8	4884	34299
53	24	440	8	5216	28311
54	200	2002	8	7586	60416
55	17	321	8	4337	31145
56	0	170	8	2741	19408
57	94	801	8	5818	48570
58	54	731	8	2199	16884
59	3877	32537	8	23980	240706
60	2660	23124	8	19810	187911
61	8	260	8	6255	39937
62	152	901	8	5484	34359
63	1	211	8	7249	39878
64	2553	22332	8	14250	128915
65	157	1271	8	1799	11596
66	35	371	8	1675	11256
67	80	1132	8	3158	21201
68	1	200	8	875	4917
69	9	200	8	2065	10115
70	23	460	8	3909	29702
71	1	171	8	1694	9995
72	669	4686	8	5832	42671
73	10	230	8	2408	14781
74	0	181	8	2608	14902
75	7	240	8	6475	42230
76	14	271	8	1521	9404

77	50	681	8	1611	10635
78	71	751	8	3068	18677
79	423	3495	8	4804	31896
80	12	250	8	1172	7681
81	18	300	8	1765	12488
82	9	250	8	2364	11737
83	86	831	8	2764	19758
84	15	321	8	5494	45526
85	35	551	8	2191	17094
86	161	1993	8	3426	27961
87	1	240	8	4998	39787
88	3	230	8	5154	37103
89	1052	10986	8	4204	35491
90	90	931	8	4565	34680
91	0	200	8	4415	28882
92	12	250	8	4017	23914
93	61	711	8	3070	24966
94	4	250	8	2633	15973
95	133	1792	8	2801	21971
96	69	441	8	2879	21812
97	130	1502	8	2606	20029
98	0	180	8	3023	23744
99	0	170	8	3410	27900
100	3479	30724	8	13601	120363
101	684	7401	8	7380	57443
102	60	821	8	5307	34710
103	52	841	8	3076	23624
104	91	1001	8	3640	21891
105	0	210	8	2833	20620
106	16	340	8	3748	23994
107	11	351	8	3796	30674
108	17	441	8	3474	24195
109	2	240	8	1864	13880
110	28	471	8	5114	31415
111	82	942	8	2976	18757
112	278	3175	8	10293	81818
113	21	410	8	3770	32477
114	0	190	8	1549	10745
115	72	741	8	1926	17295
116	0	210	8	2682	18527
117	383	4196	8	3305	28280
118	8	281	8	2381	17355
119	0	180	8	1910	11427
120	0	200	8	3291	22692
121	2	241	8	8321	52936
122	642	5308	8	8792	83310
123	6	251	8	855	4537
124	1	230	8	1840	14290

125	8	321	8	2941	24576
126	31	300	8	1491	9443
127	6	311	8	2829	16864
128	0	221	8	7715	70091
129	34	431	8	5468	30805
130	121	1582	8	2726	25847
131	4	230	8	1584	11016
132	87	781	8	1585	9453
133	0	230	8	1606	10045
134	11	290	8	1863	10965
135	18	361	8	1113	6840
136	0	190	8	5125	41770
137	21	411	8	3141	23093
138	6	231	8	1887	13280
139	67	841	8	10246	93965
140	2	220	8	1247	9223
141	0	200	8	1554	11286
142	95	1302	8	2229	15943
143	0	260	8	1175	9033
144	6	260	8	1339	9292
145	184	3094	8	3924	35020
146	0	220	8	3284	32917
147	7	251	8	2769	18437
148	119	1101	8	1183	6689
149	0	221	8	813	5488
150	123	1282	8	823	5728
151	1	201	8	694	4697
152	47	651	8	1600	11867
153	6	281	8	1196	8272
154	25	401	8	1371	7351
155	0	210	8	1492	11256
156	22	381	8	3316	26718
157	1	211	8	1379	7681
158	49	691	8	1604	11847
159	67	752	8	1453	8673
160	4	270	8	1526	11126
161	0	210	8	730	4556
162	2	250	8	3237	17324
163	10	391	8	2203	20770
164	10	511	8	641	4236
165	2	241	8	1669	11988
166	380	4386	16	2375	22392
167	380	4526	16	2375	21961
168	4	291	16	4047	33298
169	4	291	16	4047	33709
170	0	230	8	797	5758
171	0	230	8	1408	8182
172	1	240	8	1630	8602

173	2	261	8	1138	7491
174	2	240	8	1036	6960
175	1	240	8	2718	24055
176	0	230	8	1511	14290
177	1	251	8	3121	20570
178	4	330	8	2306	15061
179	65	751	8	1245	8473
180	10	410	8	831	6639
181	185	1793	8	983	7521
182	0	230	16	3233	21020
183	0	211	16	3233	21341
184	0	290	8	1128	9714
185	0	200	8	804	6679
186	0	241	8	2037	12659
187	2	240	8	583	3324
188	5	271	8	706	5138
189	75	1352	8	1744	17155
190	1	320	8	2552	17735
191	0	240	8	1160	8292
192	2	250	8	714	4897
193	275	3635	8	3277	31886
194	40	471	8	1007	6469
195	10	341	8	1788	13249
196	8	281	8	1964	13860
197	7	441	8	1273	9754
198	0	220	8	1916	10906
199	2	250	8	1384	8893
200	4	250	8	1774	15181
201	5	281	8	2132	17826
202	0	230	8	1971	10786
203	0	230	8	1704	9824
204	0	260	8	1918	13109
205	2	380	8	1261	11817
206	0	270	8	2214	14791
207	1	310	8	1389	10345

Possible Improvements

A major decrease of the total execution time is possible by introducing constraints to avoid redundant, symmetrical solutions. For each problem in the data-set, there are 8 symmetrical solutions obtained by applying the symmetries in the plane. We choose not to attempt this in this benchmarking problem, in order to simplify comparisons between different systems.

Summary

In this note we have presented the solution of a set of perfect square placement problems with CHIP. A rather simple CHIP program can be used to find solutions to all problem instances.

Bibliography

[AB93] A. Aggoun, N. Beldiceanu
Extending CHIP in Order to Solve Complex Scheduling Problems
Journal of Mathematical and Computer Modelling, Vol. 17, No. 7, pages 57 -73
Pergamon Press, 1993

[BC 94] N. Beldiceanu, E. Contejean
Introducing Global Constraints in CHIP
Journal of Mathematical and Computer Modelling, Vol 20, No 12, pp 97-123, 1994

[BD92] C.J. Bouwkamp and A.J.W. Duijvestijn
Catalogue of Simple Perfect Squared Squares of orders 21 through 25
Eindhoven University of Technology, Department of Mathematics and Computer Science, Technical Report 92 WSK 03
Eindhoven, The Netherlands, November 1992

[CFG91] H. T. Croft, K.J. Falconer, R.K. Guy
Unsolved Problems in Geometry
Springer-Verlag, New York, Berlin, 1991

[Dui78] A.J.W. Duijvestijn
Simple Perfect Squared Square of Lowest Order.
Journal of Combinatorial Theory, Series B 25, 555-558, 1978

[LW92] J. H. van Lint, R.M. Wilson
A Course in Combinatorics
Cambridge University Press, 1992

[VH94] P. Van Hentenryck
Scheduling and Packing in the Constraint Language cc(FD)
in M. Zweben, M. Fox: Intelligent Scheduling
Morgan Kaufmann Publishers, San Francisco, USA, 1994

Appendix A

The following table contains all the data for all problem instances from [BD92].
The fields describe the problem number, the number of squares, the size of the master square and a list of the square sizes. The problem number corresponds to the page number in [BD92]. Problems 166 and 167, 168 and 169, 182 and 183 are identical, but have two non-isomorphic solutions.

1,21,112,[2,4,6,7,8,9,11,15,16,17,18,19,24,25,27,29,33,35,37,42,50]
2,22,110,[2,3,4,6,7,8,12,13,14,15,16,17,18,21,22,23,24,26,27,28,50,60]
3,22,110,[1,2,3,4,6,8,9,12,14,16,17,18,19,21,22,23,24,26,27,28,50,60]
4,22,139,[1,2,3,4,7,8,10,17,18,20,21,22,24,27,28,29,30,31,32,38,59,80]
5,22,147,[1,3,4,5,8,9,17,20,21,23,25,26,29,31,32,40,43,44,47,48,52,55]
6,22,147,[2,4,8,10,11,12,15,19,21,22,23,25,26,32,34,37,41,43,45,47,55,59]
7,22,154,[2,5,9,11,16,17,19,21,22,24,26,30,31,33,35,36,41,46,47,50,52,61]
8,22,172,[1,2,3,4,9,11,13,16,17,18,19,22,24,33,36,38,39,42,44,53,75,97]
9,22,192,[4,8,9,10,12,14,17,19,26,28,31,35,36,37,41,47,49,57,59,62,71,86]
10,23,110,[1,2,3,4,5,7,8,10,12,13,14,15,16,19,21,28,29,31,32,37,38,41,44]
11,23,139,[1,2,7,8,12,13,14,15,16,18,19,20,21,22,24,26,27,28,32,33,38,59,80]
12,23,140,[1,2,3,4,5,8,10,13,16,19,20,23,27,28,29,31,33,38,42,45,48,53,54]
13,23,140,[2,3,4,7,8,9,12,15,16,18,22,23,24,26,28,30,33,36,43,44,47,50,60]

14,23,145,[1,2,3,4,6,8,9,12,15,20,22,24,25,26,27,29,30,31,32,34,36,61,84]
15,23,180,[2,4,8,10,11,12,15,19,21,22,23,25,26,32,33,34,37,41,43,45,47,88,92]
16,23,188,[2,4,8,10,11,12,15,19,21,22,23,25,26,32,33,34,37,45,47,49,51,92,96]
17,23,208,[1,3,4,9,10,11,12,16,17,18,22,23,24,40,41,60,62,65,67,70,71,73,75]
18,23,215,[1,3,4,9,10,11,12,16,17,18,22,23,24,40,41,60,66,68,70,71,74,76,79]
19,23,228,[2,7,9,10,15,16,17,18,22,23,25,28,36,39,42,56,57,68,69,72,73,87,99]
20,23,257,[2,3,9,11,14,15,17,20,22,24,28,29,32,33,49,55,57,60,63,66,79,123,134]
21,23,332,[1,15,17,24,26,30,31,38,47,48,49,50,53,56,58,68,83,89,91,112,120,123,129]
22,24,120,[3,4,5,6,8,9,10,12,13,14,15,16,17,19,20,23,25,32,33,34,40,41,46,47]
23,24,186,[2,3,4,7,8,9,12,15,16,18,22,23,24,26,28,30,33,36,43,46,47,60,90,96]
24,24,194,[2,3,7,9,10,16,17,18,19,20,23,25,28,34,36,37,42,53,54,61,65,68,69,72]
25,24,195,[2,4,7,10,11,16,17,18,21,26,27,30,39,41,42,45,47,49,52,53,54,61,63,80]
26,24,196,[1,2,5,10,11,15,17,18,20,21,24,26,29,31,32,34,36,40,44,47,48,51,91,105]
27,24,201,[1,3,4,6,9,10,11,12,17,18,20,21,22,23,26,38,40,46,50,52,53,58,98,103]
28,24,201,[1,4,5,8,9,10,11,15,16,18,19,20,22,24,26,39,42,44,49,52,54,56,93,108]
29,24,203,[1,2,5,10,11,15,17,18,20,21,24,26,29,31,32,34,36,40,44,48,54,58,98,105]
30,24,247,[3,5,6,9,12,14,19,23,24,25,28,32,34,36,40,45,46,48,56,62,63,66,111,136]
31,24,253,[2,4,5,9,13,18,20,23,24,27,28,31,38,40,44,50,61,70,72,77,79,86,88,104]
32,24,255,[3,5,10,11,16,17,20,22,23,25,26,27,28,32,41,44,52,53,59,63,65,74,118,137]
33,24,288,[2,7,9,10,15,16,17,18,22,23,25,28,36,39,42,56,57,60,68,72,73,87,129,159]
34,24,288,[1,5,7,8,9,14,17,20,21,26,30,32,34,36,48,51,54,59,64,69,72,93,123,165]
35,24,290,[2,3,8,9,11,12,14,17,21,30,31,33,40,42,45,48,59,61,63,65,82,84,124,166]
36,24,292,[1,2,3,8,12,15,16,17,20,22,24,26,29,33,44,54,57,60,63,67,73,102,117,175]
37,24,304,[3,5,7,11,12,17,20,22,25,29,35,47,48,55,56,57,69,72,76,92,96,100,116,132]
38,24,304,[3,4,7,12,16,20,23,24,27,28,30,32,33,36,37,44,53,57,72,76,85,99,129,175]
39,24,314,[2,4,11,12,16,17,18,19,28,29,40,44,47,59,62,64,65,78,79,96,97,105,113,139]
40,24,316,[3,9,10,12,13,14,15,23,24,33,36,37,48,52,54,55,57,65,66,78,79,93,144,172]
41,24,326,[1,6,10,11,14,15,18,24,29,32,43,44,53,56,63,65,71,80,83,101,104,106,119,142]
42,24,423,[2,9,15,17,27,29,31,32,33,36,47,49,50,60,62,77,105,114,123,127,128,132,168,186]
43,24,435,[1,2,8,10,13,19,23,33,44,45,56,74,76,78,80,88,93,100,112,131,142,143,150,192]
44,24,435,[3,5,9,11,12,21,24,27,30,44,45,50,54,55,63,95,101,112,117,123,134,140,178,200]
45,24,459,[8,9,10,11,16,30,36,38,45,55,57,65,68,84,95,98,100,116,117,126,135,144,180,198]
46,24,459,[4,6,9,10,17,21,23,25,31,33,36,38,45,50,83,115,117,126,133,135,144,146,180,198]
47,24,479,[5,6,17,23,24,26,28,29,35,43,44,52,60,68,77,86,130,140,150,155,160,164,174,175]
48,25,147,[3,4,5,6,8,9,10,12,13,14,15,16,17,19,20,23,25,27,32,33,34,40,41,73,74]
49,25,208,[1,2,3,4,5,7,8,11,12,17,18,24,26,28,29,30,36,39,44,45,50,59,60,89,119]
50,25,213,[3,5,6,7,13,16,17,20,21,23,24,25,26,28,31,35,36,47,49,56,58,74,76,81,90]
51,25,215,[1,4,6,7,11,15,24,26,27,33,37,39,40,41,42,43,45,47,51,55,60,62,63,69,83]
52,25,216,[1,2,3,4,5,7,8,11,16,17,18,19,25,30,32,33,39,41,45,49,54,59,64,103,113]
53,25,236,[1,2,4,9,11,12,13,14,15,16,19,24,38,40,44,46,47,48,59,64,65,70,81,85,107]
54,25,242,[1,3,6,7,9,13,14,16,17,19,23,25,26,28,30,31,47,51,54,57,60,64,67,111,131]
55,25,244,[1,2,4,5,7,10,15,17,19,20,21,22,26,27,30,37,40,41,45,65,66,68,70,110,134]
56,25,252,[4,7,10,11,12,13,23,25,29,31,32,34,36,37,38,40,42,44,62,67,68,71,77,108,113]
57,25,253,[2,4,5,6,9,10,12,14,20,24,27,35,36,37,38,42,43,45,50,54,63,66,70,120,133]
58,25,260,[1,4,6,7,10,15,24,26,27,28,29,31,33,34,37,38,44,65,70,71,77,78,83,100,112]
59,25,264,[3,7,8,12,16,18,19,20,22,24,26,31,34,37,38,40,42,53,54,61,64,69,70,130,134]
60,25,264,[3,8,12,13,16,18,20,21,22,24,26,29,34,38,40,42,43,47,54,59,64,70,71,130,134]
61,25,264,[1,3,4,6,9,10,11,12,16,17,18,20,21,22,39,42,54,56,61,66,68,69,73,129,135]
62,25,265,[1,3,4,6,9,10,11,12,16,17,18,20,21,22,39,42,54,56,62,66,68,69,74,130,135]
63,25,273,[1,4,8,10,11,12,17,19,21,22,27,29,30,33,37,43,52,62,65,86,88,89,91,96,120]
64,25,273,[1,6,9,14,16,17,18,21,22,23,25,31,32,38,44,46,48,50,54,62,65,68,78,133,140]
65,25,275,[2,3,7,13,17,24,25,31,33,34,35,37,41,49,51,53,55,60,68,71,74,81,94,100,107]
66,25,276,[1,5,8,9,11,18,19,21,30,36,41,44,45,46,47,51,53,58,63,69,71,84,87,105,120]
67,25,280,[5,6,11,17,18,20,21,24,27,28,32,34,41,42,50,53,54,55,68,78,85,88,95,97,117]
68,25,280,[2,3,7,8,14,18,30,36,37,39,44,50,52,54,56,60,63,64,65,72,75,78,79,96,106]
69,25,284,[1,2,11,12,14,16,18,19,23,26,29,37,38,39,40,42,59,68,69,77,78,97,106,109,110]
70,25,286,[1,4,5,7,10,12,15,16,20,23,28,30,32,33,35,37,53,54,64,68,74,79,80,133,153]
71,25,289,[2,3,5,8,13,14,17,20,21,32,36,41,50,52,60,61,62,68,74,76,83,87,100,102,104]
72,25,289,[2,3,4,5,7,12,16,17,19,21,23,25,29,31,32,44,57,64,65,68,72,76,84,140,149]
73,25,290,[1,2,10,11,13,14,15,17,18,28,29,34,36,38,50,56,60,69,77,80,85,91,94,111,119]
74,25,293,[5,6,11,17,18,20,21,24,27,28,32,34,41,42,50,54,55,66,68,78,85,88,95,110,130]
75,25,297,[2,7,8,9,10,15,16,17,18,23,25,26,28,36,38,43,53,60,61,68,69,77,99,137,160]
76,25,308,[1,3,4,7,10,12,13,23,25,34,37,38,39,43,44,45,62,77,79,85,87,108,113,115,116]
77,25,308,[1,5,6,7,8,9,13,16,19,28,33,36,38,43,45,48,70,71,73,84,86,102,104,120,133]
78,25,309,[7,8,14,16,23,24,25,26,31,33,34,39,48,56,59,60,62,70,76,82,92,100,101,108,117]
79,25,311,[2,7,8,9,10,15,16,17,18,23,25,26,28,36,38,43,53,60,61,68,83,91,99,151,160]
80,25,314,[1,6,7,11,16,22,26,29,32,36,38,44,51,53,64,69,70,73,74,75,85,87,101,116,128]

81,25,316,[1,3,9,12,21,26,30,33,34,35,38,39,40,41,53,56,59,69,79,85,96,103,111,117,120]
82,25,317,[1,5,6,7,8,9,16,17,19,32,37,40,42,47,49,52,59,75,81,92,94,110,112,113,126]
83,25,320,[2,7,8,9,12,14,15,21,23,35,38,44,46,49,53,54,56,63,96,101,103,105,108,112,116]
84,25,320,[3,8,9,11,17,18,22,25,26,27,29,30,31,33,35,49,51,67,72,73,80,85,95,152,168]
85,25,320,[1,4,6,7,8,13,14,16,24,28,30,33,34,38,41,42,57,60,69,78,81,90,92,150,170]
86,25,320,[3,4,6,8,9,14,15,16,24,28,30,31,34,38,39,42,59,60,71,78,79,90,92,150,170]
87,25,322,[3,4,8,9,10,16,18,20,22,23,24,28,31,38,44,47,64,65,68,76,80,81,97,144,178]
88,25,322,[3,4,8,10,15,16,18,19,20,22,24,28,35,38,44,53,59,64,68,76,80,85,93,144,178]
89,25,323,[2,3,4,7,10,13,15,18,23,32,34,35,36,42,46,50,57,60,66,72,78,87,98,159,164]
90,25,323,[3,8,9,11,17,18,22,25,26,27,29,30,31,33,35,49,51,67,72,73,83,88,95,155,168]
91,25,323,[2,6,9,11,13,14,18,19,20,23,27,28,29,42,46,48,60,64,72,74,79,82,98,146,177]
92,25,325,[3,5,6,11,12,13,18,23,25,28,32,37,40,43,45,46,51,79,92,99,103,108,112,114,134]
93,25,326,[1,4,8,10,12,16,21,22,24,27,28,35,36,37,38,46,49,68,70,75,88,90,93,158,168]
94,25,327,[2,9,10,12,13,16,19,21,23,26,36,44,46,52,55,61,62,74,84,87,100,103,104,120,140]
95,25,328,[2,3,4,7,8,10,14,17,26,27,28,36,38,40,42,45,53,58,73,74,79,94,102,152,176]
96,25,334,[1,4,8,10,12,16,21,22,24,27,28,35,36,37,38,46,49,68,75,78,88,93,98,166,168]
97,25,336,[2,3,4,7,8,10,14,17,26,27,28,36,38,40,45,50,53,58,73,74,79,94,110,152,184]
98,25,338,[1,4,8,10,12,16,19,22,24,25,28,36,37,38,39,46,53,68,70,73,94,96,101,164,174]
99,25,338,[4,5,8,10,12,15,16,21,22,24,28,33,36,38,43,46,57,68,70,77,94,96,97,164,174]
100,25,340,[1,4,5,6,11,13,16,17,22,24,44,46,50,51,52,53,61,64,66,79,84,85,92,169,171]
101,25,344,[2,3,8,11,14,17,19,21,23,25,27,36,39,44,48,53,56,71,77,83,86,89,98,169,175]
102,25,359,[7,8,9,10,14,17,18,23,25,27,29,31,40,41,43,46,69,74,82,85,90,98,102,172,187]
103,25,361,[2,6,7,8,9,14,20,22,26,27,32,34,36,47,49,56,66,67,74,82,89,98,107,156,205]
104,25,363,[1,4,6,12,13,20,21,25,26,27,28,32,37,41,45,53,58,64,69,91,97,102,106,155,208]
105,25,364,[2,3,4,6,8,9,13,14,16,19,23,24,28,29,52,57,64,75,82,91,98,100,109,173,191]
106,25,367,[1,4,6,12,13,20,21,25,26,27,28,32,37,41,49,53,58,64,69,91,97,102,110,155,212]
107,25,368,[1,6,15,16,17,18,22,25,31,33,39,42,45,46,47,48,51,69,72,88,91,96,112,160,208]
108,25,371,[1,2,7,8,20,21,22,24,26,28,30,38,43,46,50,51,64,65,70,90,95,102,109,160,211]
109,25,373,[3,6,7,8,15,17,22,23,31,32,35,41,43,60,62,68,79,87,104,105,114,120,121,138,148]
110,25,378,[2,3,10,17,18,20,21,22,24,27,31,38,41,48,51,56,68,78,80,85,87,96,117,165,213]
111,25,378,[1,2,7,13,15,17,18,25,27,29,30,31,42,43,46,56,61,68,73,93,100,105,112,161,217]
112,25,380,[4,7,17,18,19,20,21,26,31,33,35,40,45,48,49,60,67,73,79,81,87,107,113,186,194]
113,25,380,[4,5,6,9,13,15,16,17,22,24,33,38,44,49,50,56,60,67,82,84,95,108,121,177,203]
114,25,381,[12,13,21,23,25,27,35,36,42,45,54,57,59,60,79,82,84,85,92,95,96,100,110,111,186]
115,25,384,[1,4,8,9,11,12,19,21,27,32,35,44,45,46,47,51,60,67,84,89,96,108,120,180,204]
116,25,384,[1,4,8,9,11,12,15,17,19,25,26,31,32,37,44,57,60,81,84,96,99,108,120,180,204]
117,25,384,[3,5,7,11,12,17,20,22,25,29,35,47,48,55,56,57,69,72,76,80,96,100,116,172,212]
118,25,385,[1,2,7,13,15,17,18,25,27,29,30,31,43,46,49,56,61,68,73,93,100,105,119,161,224]
119,25,392,[4,7,8,15,23,26,29,30,31,32,34,43,48,55,56,68,77,88,98,106,116,135,141,151,153]
120,25,392,[10,12,14,16,19,21,25,27,31,35,39,41,51,52,54,55,73,92,98,115,121,123,129,148,171]
121,25,392,[1,4,5,8,11,14,16,21,22,24,27,28,30,31,52,64,81,83,96,97,98,99,114,195,197]
122,25,393,[4,8,16,20,23,24,25,27,29,37,44,45,50,53,64,66,68,69,73,85,91,101,116,186,207]
123,25,396,[1,4,5,14,16,32,35,36,46,47,48,49,68,69,73,93,94,97,99,104,110,111,125,126,160]
124,25,396,[1,4,5,8,11,14,16,21,22,24,27,28,30,31,52,64,81,83,98,99,100,101,114,197,199]
125,25,396,[3,8,9,11,14,16,17,18,31,32,41,45,48,56,60,66,73,75,81,82,98,99,117,180,216]
126,25,398,[2,6,7,11,15,17,23,28,29,39,44,46,53,56,58,65,68,99,100,119,120,134,144,145,154]
127,25,400,[3,6,21,23,24,26,29,35,37,40,41,47,53,55,64,76,79,81,99,100,121,122,137,142,179]
128,25,404,[3,6,7,14,17,20,21,26,28,31,32,39,46,53,54,68,71,80,88,92,100,111,113,199,205]
129,25,404,[4,7,10,11,12,13,16,18,20,23,25,28,29,32,47,62,70,88,93,96,101,114,127,189,215]
130,25,408,[2,3,7,13,16,18,20,27,30,33,41,43,46,52,54,57,72,79,84,100,105,108,116,195,213]
131,25,412,[3,11,12,15,21,26,32,39,43,47,54,60,68,73,83,85,86,87,89,99,114,129,139,144,169]
132,25,413,[5,7,17,20,34,38,39,48,56,57,59,60,64,65,70,72,75,81,105,106,110,125,148,153,155]
133,25,416,[2,4,7,11,13,24,25,30,35,37,39,40,44,58,62,65,82,104,112,120,128,135,143,153,169]
134,25,416,[1,2,3,8,12,15,16,17,20,22,24,26,29,31,64,75,85,88,91,94,98,104,133,179,237]
135,25,421,[1,2,4,5,7,9,12,16,20,22,23,35,38,48,56,83,94,104,116,118,128,140,150,153,177]
136,25,421,[5,11,12,17,18,20,23,26,29,36,38,40,44,51,55,59,72,92,97,102,105,107,117,199,222]
137,25,422,[2,4,7,13,16,18,20,23,28,29,38,43,46,51,59,68,74,79,86,93,100,111,132,179,243]
138,25,425,[3,4,5,9,10,12,13,14,16,19,20,31,46,48,56,79,102,104,116,126,128,140,142,157,181]
139,25,441,[5,6,7,16,18,23,24,27,38,39,47,51,52,62,66,72,80,84,92,101,102,118,120,219,222]
140,25,454,[1,2,11,17,29,34,35,46,48,51,53,55,63,69,79,87,88,91,109,134,136,143,150,161,184]
141,25,456,[5,7,10,11,13,15,18,19,31,49,50,52,59,60,63,72,77,115,128,129,135,142,148,179,193]
142,25,465,[6,9,13,14,19,21,24,25,31,32,53,56,64,73,74,82,91,111,125,127,137,139,153,173,201]
143,25,472,[7,9,13,15,26,34,35,44,47,51,58,61,65,81,87,103,104,115,118,123,128,133,136,148,221]
144,25,477,[3,5,12,16,19,22,25,26,37,41,49,72,76,77,82,86,87,115,117,135,141,149,167,169,193]
145,25,492,[2,9,15,17,27,29,31,32,33,36,47,49,50,60,62,69,77,105,114,123,127,128,132,237,255]
146,25,492,[3,5,9,11,12,21,24,27,30,44,45,50,54,55,57,63,95,101,112,117,123,134,140,235,257]
147,25,503,[4,15,16,19,22,23,25,27,33,34,50,62,67,87,88,93,100,113,135,143,149,157,167,179,211]

148,25,506,[1,7,24,26,33,35,40,45,47,51,55,69,87,90,93,96,117,125,134,145,146,147,160,162,199]
149,25,507,[2,3,7,11,13,15,28,34,43,50,57,64,80,83,86,89,107,115,116,127,149,163,175,183,217]
150,25,512,[1,7,8,9,10,15,22,32,34,46,51,65,69,71,91,105,109,111,136,139,152,157,173,200,203]
151,25,512,[1,6,7,8,9,13,17,19,35,45,47,57,62,73,88,93,104,107,128,130,151,163,184,198,221]
152,25,513,[6,9,10,17,19,24,28,29,37,39,64,65,68,81,98,99,102,115,145,147,153,159,165,189,201]
153,25,517,[5,6,7,16,20,24,28,33,38,43,63,71,80,83,86,92,98,122,132,148,164,166,173,180,205]
154,25,524,[9,12,20,21,33,35,37,39,54,55,61,62,87,90,98,101,125,132,135,141,145,159,163,164,220]
155,25,527,[11,12,13,14,19,30,41,47,50,52,59,68,71,81,94,97,107,132,147,151,155,169,175,183,197]
156,25,528,[2,9,15,17,27,29,31,32,33,36,47,49,50,60,62,69,77,123,127,128,132,141,150,255,273]
157,25,529,[9,12,20,21,33,35,37,39,54,55,61,62,87,90,98,101,125,132,140,141,145,159,163,169,225]
158,25,531,[6,9,10,17,19,24,29,31,39,40,67,68,71,84,101,102,105,118,151,153,159,165,171,195,207]
159,25,532,[16,18,26,27,33,39,41,50,51,55,69,71,84,87,91,94,132,133,141,143,164,168,169,173,195]
160,25,534,[11,13,15,17,18,27,38,44,49,52,60,61,68,81,87,94,107,135,149,153,159,171,174,189,210]
161,25,535,[2,8,26,27,36,41,45,57,62,77,88,95,97,99,101,102,109,114,117,118,141,147,168,192,226]
162,25,536,[1,8,21,30,31,32,33,41,44,46,49,55,57,61,84,91,113,134,137,139,150,155,176,205,247]
163,25,536,[3,5,9,11,12,21,24,27,30,44,45,50,54,55,57,63,95,117,123,134,140,145,156,257,279]
164,25,540,[1,7,8,9,10,14,19,34,36,51,58,69,81,83,97,109,111,115,136,149,152,167,183,208,221]
165,25,540,[6,13,15,25,28,36,43,47,55,57,58,59,60,65,82,89,91,107,124,127,144,163,183,233,250]
166,25,540,[8,9,10,11,16,30,36,38,45,55,57,65,68,81,84,95,98,100,116,117,126,135,144,261,279]
167,25,540,[8,9,10,11,16,30,36,38,45,55,57,65,68,81,84,95,98,100,116,117,126,135,144,261,279]
168,25,540,[4,6,9,10,17,21,23,25,31,33,36,38,45,50,81,83,115,117,126,133,135,144,146,261,279]
169,25,540,[4,6,9,10,17,21,23,25,31,33,36,38,45,50,81,83,115,117,126,133,135,144,146,261,279]
170,25,541,[3,4,11,13,16,17,21,25,26,44,46,64,75,86,87,97,106,109,133,141,165,185,191,215,217]
171,25,541,[3,5,27,32,33,37,47,50,53,56,57,69,71,78,97,98,109,111,126,144,165,169,183,189,232]
172,25,544,[1,7,24,26,33,35,40,45,47,51,55,69,87,90,93,96,117,125,134,145,147,184,198,199,200]
173,25,544,[6,8,20,21,23,41,42,48,59,61,77,80,81,85,90,92,93,102,115,132,139,168,198,207,244]
174,25,547,[3,5,16,22,26,27,35,47,49,59,67,71,72,85,87,102,103,111,137,144,150,197,200,203,207]
175,25,549,[4,10,14,24,26,31,34,36,38,40,43,48,59,63,74,89,97,105,117,124,136,152,156,241,308]
176,25,550,[1,2,5,13,19,20,25,30,39,43,58,59,73,75,76,90,95,103,116,128,130,132,172,262,288]
177,25,550,[1,11,16,23,24,27,29,36,41,43,44,47,59,70,71,80,99,103,111,116,128,156,167,227,323]
178,25,551,[3,5,24,25,26,30,35,36,39,40,42,57,68,76,94,109,120,128,152,162,166,175,176,200,223]
179,25,552,[5,17,18,22,25,27,32,33,39,59,62,87,91,100,102,111,112,135,137,149,165,168,183,201,204]
180,25,552,[1,3,4,7,8,9,10,15,18,19,21,41,52,54,73,93,95,123,125,136,138,153,168,261,291]
181,25,556,[6,8,10,13,19,25,32,37,49,54,58,76,84,91,92,100,107,128,145,156,165,185,195,205,206]
182,25,556,[3,12,13,15,19,23,27,34,35,39,42,45,48,52,53,87,140,145,158,166,171,184,189,201,227]
183,25,556,[3,12,13,15,19,23,27,34,35,39,42,45,48,52,53,87,140,145,158,166,171,184,189,201,227]
184,25,556,[1,5,7,8,9,10,12,14,20,27,31,43,47,50,74,93,97,121,125,139,143,153,167,264,292]
185,25,562,[2,3,5,8,13,19,20,29,33,47,53,54,64,65,76,93,119,123,142,157,161,180,184,221,259]
186,25,570,[3,9,10,33,36,38,40,42,50,51,60,69,72,75,77,90,113,140,141,151,152,189,200,229,230]
187,25,575,[4,6,14,16,31,39,63,69,74,81,88,103,107,111,115,120,131,132,133,147,156,159,164,198,218]
188,25,576,[1,4,9,11,15,19,22,34,36,53,60,76,82,84,104,126,127,128,153,156,165,174,183,219,237]
189,25,576,[8,9,10,11,16,30,36,38,45,55,57,65,68,81,84,95,98,100,116,135,144,153,162,279,297]
190,25,576,[4,6,9,10,17,21,23,25,31,33,36,38,45,50,81,83,115,133,135,144,146,153,162,279,297]
191,25,580,[2,5,7,10,12,13,19,21,22,29,36,40,61,65,74,101,135,139,161,179,183,192,205,209,236]
192,25,580,[5,6,11,13,16,17,21,25,34,44,54,68,80,88,100,112,120,135,142,145,170,173,195,215,265]
193,25,580,[11,12,16,17,29,32,39,41,53,55,59,60,68,70,81,84,92,124,125,128,129,156,171,280,300]
194,25,593,[13,14,15,35,48,51,55,67,73,79,83,91,94,105,109,116,119,124,133,150,171,173,196,217,226]
195,25,595,[4,13,18,19,22,35,40,48,58,61,62,77,78,82,83,86,118,149,163,168,187,192,202,206,240]
196,25,601,[7,8,25,34,41,42,46,48,54,55,62,70,71,74,98,103,116,143,168,169,190,192,193,218,240]
197,25,603,[7,11,12,14,21,25,32,40,52,56,60,67,68,81,91,92,132,144,149,163,177,191,196,235,263]
198,25,603,[13,23,26,27,35,44,45,49,53,54,57,66,75,99,101,110,122,126,144,158,175,180,189,234,270]
199,25,607,[6,8,10,13,19,25,32,37,49,54,58,76,84,91,92,100,107,128,156,185,196,205,206,216,246]
200,25,609,[9,14,15,17,32,45,47,58,67,74,76,79,80,83,97,111,125,126,150,170,186,188,215,224,235]
201,25,611,[1,10,22,26,32,41,45,54,57,61,62,66,85,86,87,95,97,101,119,132,136,167,176,268,343]
202,25,614,[15,22,24,31,33,49,53,54,57,60,63,68,74,81,83,104,109,151,155,163,167,217,229,230,234]
203,25,634,[15,17,24,26,33,43,44,54,57,60,63,73,79,81,88,109,119,160,161,172,173,227,234,235,239]
204,25,643,[2,9,21,29,38,40,41,42,58,62,67,76,82,83,85,96,104,166,172,186,192,201,207,250,270]
205,25,644,[7,9,13,18,19,22,31,49,53,61,66,68,71,87,93,94,119,164,178,192,199,206,227,239,253]
206,25,655,[10,14,15,21,25,26,31,40,51,53,54,57,65,83,84,86,151,152,173,193,194,215,216,246,288]
207,25,661,[5,7,17,18,23,31,36,38,41,64,73,77,83,84,102,106,111,161,175,196,203,210,238,248,262]